CELLULAR BASES FOR ALGEBRAS WITH A JONES BASIC CONSTRUCTION

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Contents

Abstract. We define a method which produces explicit cellular bases for algebras obtained via a Jones basic construction. For the class of algebras in question, our method gives formulae for generic Murphy–type cellular bases indexed by paths on branching diagrams and compatible with restriction and induction on cell modules. The construction given here allows for a uniform combinatorial treatment of cellular bases and representations of the Brauer, Birman–Murakami–Wenzl, Temperley–Lieb, and partition algebras, among others.

1. Introduction

The notion of cellularity was introduced by Graham and Lehrer [GL] as a tool for studying non–semisimple representations of Hecke algebras and other algebras with geometric connections. Cellular algebras are defined by the existence of a *cellular basis* with combinatorial properties that reflect the "Robinson–Schensted correspondence" in the Iwahori–Hecke algebra of the symmetric group. Important examples of cellular algebras include the Iwahori–Hecke algebras of the symmetric group, Brauer algebras, Birman–Murakami–Wenzl algebras, Temperley–Lieb algebras and partition algebras (see [GL], [Mu], [Xi], [Xi1]).

In this paper, we consider a tower of unital algebras

$$R = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \tag{1.1}$$

over an integral domain R which are obtained by a repeated Jones basic construction on a tower of cellular algebras

$$R = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \tag{1.2}$$

with cellular bases that are well behaved with respect to restriction and induction on cell modules. For such pairs of towers, we demonstrate:

(1) An explicit filtration of each cell module for A_{i+1} by cell modules for A_i and;

(2) An inductive construction by which a cellular basis for A_{i+1} is explicitly defined in terms of a cellular basis for A_i.

Key words and phrases. Cellular algebra; Jones basic construction; Murphy basis; Brauer algebra, Birman–Murakami–Wenzl algebra; Partition algebra.

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Having established an inductive construction of cellular bases in the above general setting, we apply our method obtain explicit cellular bases for the Brauer algebras, Birman–Murakami–Wenzl (BMW) algebras, Temperley–Lieb algebras, and partition algebras. In each of the aforementioned examples, we write down explicit formulae for cellular bases that are indexed by paths on an appropriate branching diagram. The given bases are compatible with restriction and induction on cell modules, and are analogues of Murphy's cellular bases for the Iwahori–Hecke algebra of the symmetric group [Mu]. In the particular cases of the Brauer and BMW algebras, our results recover the construction given in [En1]. Hence this work may be regarded as a generalisation of [En1].

The hypotheses which allow for an inductive construction of Murphy–type bases in this paper differ only slightly from the framework for cellularity of algebras related to the Jones basic construction given by Goodman and Graber in [GG] and, in each of the examples we consider, the existence of a Murphy–type cellular basis which is compatible with induction and restriction has already been established non–explicitly in [GG].

Given that our bases are compatible with restriction and induction on cell modules, the Jucys–Murphy elements in each algebra under consideration here will act triangularly relative to the Murphy–type bases given here (see §3 of [GG1] or §3 of [Mat1]). Thus, in the generic setting, and after a unitriangular transformation, the Murphy–type cellular bases will give seminormal bases of each of the algebras under consideration here. In [En2], the Murphy–type bases given in §4 are used to explicit combinatorial formulae for the seminormal representations of the partition algebras.

Using the definition of the partition algebras as diagram algebras, Xi [Xi] has given cellular bases for the partition algebras (see also [DW], [Wi]). The bases for partition algebras given in [Xi] are obtained by adjoining certain tangle diagrams to basis elements for the the group algebra of the symmetric group, a process which formally corresponds to König and Xi's method of constructing cellular algebras by inflation [KX]. The method established in [Xi] has also been used to give prove cellularity of bases for the Brauer and BMW algebras which are indexed by tangle diagrams (see [Xi1], [En], [Wi]). The significance of the approach taken in this paper is that by indexing the basis elements by paths in suitably constructed branching diagrams, rather than by tangles, we are able to obtain cellular structures which admit explicit cell module filtrations under induction and restriction.

Finally, we note that the results of Ariki and Mathas [AM] and Mathas [Mat2] on restriction and induction on cell modules of the cyclotomic hecke algebras, imply that the construction of cellular bases given here applies equally well to the cyclotomic BMW algebras with admissible parameters. In this setting, our construction would recover the generalisation of [En1] to the cyclotomic case given by Rui and Xu in [RX].

In §2 we recall the definition of cellularity from [GL] and [GG]. We define the branching diagram of a tower of cellular algebras in terms of restriction and induction on cell modules and formulate a set of hypotheses on the towers (1.1) and (1.2), analogous to the framework for cellularity given by Goodman and Graber [GG]. In §3 we construct explicit filtration which show that restriction and induction on certain modules in the tower (1.1) are controlled by restriction and induction on cell modules in the tower (1.2). In §4 we show that (1.1) is a tower of cellular algebras with Murphy type bases that are well behaved with respect to restriction and induction on cell modules. In §5 we apply the preceding construction to particular examples.

2. Preliminiaries

Cellular algebras were defined by Graham and Lehrer [GL]. The construction in this paper will use a slightly weaker version of cellularity which is due to Goodman and Graber [GG].

Definition 2.1. Let R be an integral domain. A *cellular algebra* is a tuple $(A, *, \hat{A}, \trianglerighteq, \mathscr{A})$ where

(1) A is a unital R–algebra and $* : A \to A$ is an algebra anti–automorphism of A;
(2) $(\hat{A}, \trianglerighteq)$ is an ordered set, and $\hat{A}^\lambda$, for $\lambda \in \hat{A}$, is an indexing set;
(3) The set

$$\mathscr{A} = \{c_{\mathfrak{st}}^\lambda \mid \lambda \in \hat{A} \text{ and } \mathfrak{s}, \mathfrak{t} \in \hat{A}^\lambda\},$$

is an R–basis for A, for which the following conditions hold:
(a) Given $\lambda \in \hat{A}$, $\mathfrak{t} \in \hat{A}^\lambda$, and $a \in A$, there exist $r_\mathfrak{v}$, for $\mathfrak{v} \in \hat{A}^\lambda$, such that, for all $\mathfrak{s} \in \hat{A}^\lambda$,

$$c_{\mathfrak{st}}^\lambda a \equiv \sum_{\mathfrak{v} \in \hat{A}^\lambda} r_\mathfrak{v} c_{\mathfrak{sv}}^\lambda \mod A^{\rhd \lambda}, \tag{2.1}$$

where $A^{\rhd\lambda}$ is the R–module generated by

$$\{c_{\mathfrak{st}}^\mu \mid \mu \in \hat{A},\ \mathfrak{s}, \mathfrak{t} \in \hat{A}^\mu \text{ and } \mu \rhd \lambda\}.$$

(b) If $\lambda \in \hat{A}$ and $\mathfrak{s}, \mathfrak{t} \in \hat{A}^\lambda$, then $(c_{\mathfrak{st}}^\lambda)^* \equiv (c_{\mathfrak{ts}}^\lambda) \mod A^{\rhd\lambda}$.

The tuple $(A, *, \hat{A}, \trianglerighteq, \mathscr{A})$ is a *cell datum* for A.

If A is an algebra with cell datum $(A, *, \hat{A}, \trianglerighteq, \mathscr{A})$ we will frequently omit reference to the cell datum for A and simply refer to A as a *cellular algebra*.

Let A be a cellular algebra, $\lambda \in \hat{A}$ and $\mathfrak{s} \in \hat{A}^\lambda$. The *cell module* A^λ is the R–submodule of $A/A^{\rhd\lambda}$ generated by

$$\{c_{\mathfrak{st}}^\lambda + A^{\rhd\lambda} \mid \mathfrak{t} \in \hat{A}^\lambda\}$$

with right A–action given by (2.1). By the condition (a) in Definition 2.1, A^λ is independent of the choice of $\mathfrak{s}$.

If A is a cellular algebra over R, $\lambda \in \hat{A}$, and $N \subseteq M$ is an inclusion of right A–modules, write

$$N \overset{\lambda}{\subseteq} M \qquad \text{if} \qquad M/N \cong A^\lambda \qquad \text{as right } A\text{–modules.}$$

If M is a right A–module, an order preserving A cell–module composition series for M is a filtration

$$\{0\} = M_0 \overset{\lambda^{(1)}}{\subseteq} M_1 \overset{\lambda^{(2)}}{\subseteq} \cdots \overset{\lambda^{(r)}}{\subseteq} M_r = M, \qquad (\lambda^{(1)}, \dots, \lambda^{(r)} \in \hat{A}),$$

by right A–modules, such that $\lambda^{(s)} \rhd \lambda^{(t)}$ in $\hat{A}$ whenever $t > s$.

If $A \subseteq B$ is an inclusion of algebras over R, define the induced module

$$\mathrm{Ind}_A^B(M) = M \otimes_A B.$$

Definition 2.2. Let R be an integral domain. A *tower of cellular algebras with a branching diagram* is a sequence of cellular algebras over R

$$\{(H_i, *, \hat{H}_i, \trianglerighteq, \mathscr{H}_i) \mid i = 0, 1, \dots\}$$

such that $H_0 = R$, and for $i = 0, 1, \dots,$ the following conditions hold:

(1) $H_i \subseteq H_{i+1}$ and $1_{H_i} = 1_R$.
(2) If $\lambda \in \hat{H}_i$, then there exist right H_{i+1}–modules $N_1, \dots, N_p$ and $\mu^{(1)}, \dots, \mu^{(p)} \in \hat{H}_{i+1}$, for which

$$\{0\} = N_0 \overset{\mu^{(1)}}{\subseteq} N_1 \overset{\mu^{(2)}}{\subseteq} \cdots \overset{\mu^{(p)}}{\subseteq} N_p = \mathrm{Ind}_{H_i}^{H_{i+1}}(H_i^\lambda), \tag{2.2}$$

is an order preserving cell module composition series.

(3) If $\mu \in \hat{H}_{i+1}$, then there exist right H_i–modules $M_1, \ldots, M_r$ and $\lambda^{(1)}, \ldots, \lambda^{(r)} \in \hat{H}_i$, for which

$$\{0\} = M_0 \overset{\lambda^{(1)}}{\subseteq} M_1 \overset{\lambda^{(2)}}{\subseteq} \cdots \overset{\lambda^{(r)}}{\subseteq} M_r = \operatorname{Res}_{H_i}^{H_{i+1}}\left(H_{i+1}^{\mu}\right) \tag{2.3}$$

is an order preserving cell module composition series.

(4) If $\lambda \in \hat{H}_i$ and $\mu \in \hat{H}_{i+1}$, then H_{i+1}^{μ} appears as a subquotient

$$H_{i+1}^{\mu} = N_j/N_{j-1} \qquad\qquad \text{for some } j = 1, \ldots, p,$$

in the filtration (2.2) of $\operatorname{Ind}_{H_i}^{H_{i+1}}\left(H_i^{\lambda}\right)$ if and only if H_i^{λ} appears as a subquotient

$$H_i^{\lambda} = M_j/M_{j-1} \qquad\qquad \text{for some } j = 1, \ldots, r,$$

in the filtration (2.3) of $\operatorname{Res}_{H_i}^{H_{i+1}}\left(H_{i+1}^{\mu}\right)$.

In the context of Definition 2.2 the *branching diagram* of the tower $R = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ is the graph $\hat{H}$ with

(1) vertices on level i: $\hat{H}_i$,

(2) an edge $\lambda \to \mu$, for $\lambda \in \hat{H}_i$ and $\mu \in \hat{H}_{i+1}$, if H_{i+1}^{μ} appears as a subquotient $H_{i+1}^{\mu} = N_j/N_{j-1}$ in the filtration (2.2) of $\operatorname{Ind}_{H_i}^{H_{i+1}}\left(H_i^{\lambda}\right)$ by H_{i+1}–modules.

A prototypical example of a tower of cellular algebras with a branching diagram is the tower of Iwahori–Hecke algebras of the symmetric group (see §5.2).

For the remainder of this section, we assume the following axioms which may be regarded as an adaptation to our setting of the *framework for cellularity* given by Goodman and Graber in §2.4 of [GG].

Let R be an integral domain and

$$R = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \qquad \text{and} \qquad R = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots, \tag{2.4}$$

be two towers of R–algebras each with a common multiplicative identity. We assume that:

(A) There is an algebra anti–automorphism $*$ on $\cup_i A_i$.

(B) $A_1 = H_1$ as algebras with anti–automorphisms.

(C) If $i \geqslant 2$, then A_i contains an element e_{i-1} such that $(e_{i-1})^* = e_{i-1}$ and $H_i = A_i/(A_i e_{i-1} A_i)$.

(D) For $i \geqslant 1$, e_i commutes with A_{i-1} and $e_i A_i e_i \subseteq A_{i-1} e_i$.

(E) For $i \geqslant 1$, $A_{i+1} e_i = A_i e_i$, and the map $a \mapsto a e_i$ is injective from A_i to $A_i e_i$.

(F) For $i \geqslant 1$, $e_{i+1} e_i e_{i+1} = e_{i+1}$ and $e_i e_{i+1} e_i = e_i$.

(G) $\{(H_i, *, \hat{H}_i, \trianglerighteq, \mathscr{H}_i) \mid i \geqslant 0\}$ is a tower of cellular algebras with a branching diagram, with the anti–automorphism $* : H_i \to H_i$, for $i \geqslant 2$, inherited from $* : A_i \to A_i$ via the map $A_i \to H_i$.

(H) If $i \geqslant 1$ and $\lambda \in \hat{H}_i$, then there exists $c_{\lambda}^{(i)} \in H_i$ such that $* : c_{\lambda}^{(i)} \mapsto c_{\lambda}^{(i)} \mod H_i^{\rhd\lambda}$, and

$$H_i^{\lambda} \cong \left\{c_{\lambda}^{(i)} h + H_i^{\rhd\lambda} \mid h \in H_i\right\}$$

as right H_i–modules.

In the context of hypotheses (A)–(H) above, the reader may find it useful to bear in mind the tower of Brauer algebras $R = B_0 \subseteq B_1 \subseteq \cdots$ where B_k, for $k \in \mathbb{Z}_{\geqslant 2}$, is obtained by adjoining an element e_{k-1} to the group algebra of the symmetric group on k letters (see §5.3).

For $i = 0, 1, \ldots$, let

$$\hat{A}_i = \Big\{(\lambda, \ell) \,\Big|\, \lambda \in \hat{H}_{i-2\ell}, \text{ for } \ell = 0, 1, \ldots, \lfloor i/2 \rfloor\Big\}$$

and order $\hat{A}_i$ by writing $(\lambda, \ell) \trianglerighteq (\mu, m)$, for $(\lambda, \ell), (\mu, m) \in \hat{A}_i$, if either:

(1) $\ell > m$, or

(2) $\ell = m$ and $\lambda \trianglerighteq \mu$ as elements of $\hat{H}_{i-2\ell}$.

Let $\hat{A}$ denote the graph with:

(1) vertices on level k: elements of

$$A_k = \{(\mu, m) \mid \mu \in \hat{H}_m, \text{ and } m = 0, 1, \dots, \lfloor k/2 \rfloor\},$$

(2) an edge $(\lambda, \ell) \to (\mu, m)$, for $(\lambda, \ell) \in \hat{A}_{k-1}$ and $(\mu, m) \in \hat{A}_k$, if either
  (a) $\ell = m$ and there is an edge $\lambda \to \mu$ from level $k - 2m - 1$ to level $k - 2m$ in $\hat{H}$, or
  (b) $\ell = m - 1$ and there is an edge $\mu \to \lambda$ from level $k - 2m$ to level $k - 2m + 1$ in $\hat{H}$.

If $i = 2, 3, \dots,$ let

$$e^{(\ell)}_{i-1} = \underbrace{e_{i-2\ell+1} e_{i-2\ell+3} \cdots e_{i-1}}_{\ell \text{ factors}} \qquad \text{if } \ell = 0, 1, \dots, \lfloor i/2 \rfloor,$$

and write

$$e^{(\ell)}_{i-1} = 0 \qquad \text{if } \ell > \lfloor i/2 \rfloor.$$

For $i = 0, 1, \dots,$ and $\lambda \in \hat{H}_i$, fix

$$\bar{c}^{(i)}_\lambda \in A_i \qquad \text{such that} \qquad \bar{c}^{(i)}_\mu + A_i e_{i-1} A_i \mapsto c^{(i)}_\lambda$$

under the isomorphism $A_i/(A_i e_{i-1} A_i) \to H_i$. For $(\lambda, \ell) \in \hat{A}_i$, let

$$x^{(i)}_{(\lambda,\ell)} = \bar{c}^{(i-2\ell)}_\lambda e^{(\ell)}_{i-1}$$

and define the two–sided ideal

$$A_i^{\rhd(\lambda,\ell)} = \sum_{(\mu,m) \rhd (\lambda,\ell)} A_i x^{(i)}_{(\mu,m)} A_i, \tag{2.5}$$

where the last sum is taken over $(\mu, m) \in \hat{A}_i$ such that $(\mu, m) \rhd (\lambda, \ell)$. Define the right A_i–module

$$A_i^{(\lambda,\ell)} = \left\{ x^{(i)}_{(\lambda,\ell)} a + A_i^{\rhd(\lambda,\ell)} \;\middle|\; a \in A_i \right\} \subseteq A_i / A_i^{\rhd(\lambda,\ell)}. \tag{2.6}$$

There is no loss in identifying the right A_i–modules

$$A_i^{(\lambda,0)} = H_i^\lambda \qquad \text{for } \lambda \in \hat{H}_i \text{ and } i = 1, 2, \dots.$$

Define the two sided ideal $A_i^{(\ell)} \subseteq A_i$ by

$$A_i^{(\ell)} = \begin{cases} A_i e^{(\ell)}_{i-1} A_i, & \text{if } i = 0, 1, \dots, \lfloor i/2 \rfloor, \\ \{0\} \subseteq A_i, & \text{otherwise,} \end{cases}$$

and let

$$H_i^{(\ell)} = A_i^{(\ell)} / A_i^{(\ell+1)}, \qquad \text{for } \ell = 0, 1, \dots.$$

For brevity, we will continue to write $H_i = H_i^{(0)}$. Since $A_k^{(\lambda,\ell)} A_i^{(\ell+1)} = 0$ for all $(\lambda, \ell) \in \hat{A}_i$, it makes sense to regard each $A_i^{(\lambda,\ell)}$ as a right $H_i^{(\ell)}$–module. For $i = 1, 2, \dots,$ and $\ell = 0, 1, \dots,$ let

$$R_i^{(\ell)} = \{ e^{(\ell)}_{i-1} a + A_i^{(\ell+1)} \mid a \in A_i \} \subseteq A_i / A_i^{(\ell+1)}.$$

Propositions 2.3 and 2.4 reformulate the definition of the module $A_i^{(\lambda,\ell)}$, for $(\lambda, \ell) \in \hat{A}_i$ and $\ell > 0$.

Proposition 2.3. *Let $i = 1, 2, \dots,$ and $\ell = 1, 2, \dots, \lfloor i/2 \rfloor$. Then $R_i^{(\ell)}$ is an $(H_{k-2\ell}, H_i^{(\ell)})$–bimodule, and if $(\lambda, \ell) \in \hat{A}_i$, then*

$$A_i^{(\lambda,\ell)} \cong H^\lambda_{i-2\ell} \otimes_{H_{i-2\ell}} R_i^{(\ell)}$$

as right A_i–modules.

Proof. For $h \in H_{i-2\ell}$, fix $\bar{h} \in A_{i-2\ell-1}$ such that $\bar{h} + (e_{i-2\ell}) \mapsto h$ under the isomorphism $A_{i-2\ell}/(e_{i-2\ell-1}) \to H_{i-2\ell}$. If $h \in H_{i-2\ell}$ and $a \in A_i$, then

$$\bar{h}e_{i-1}^{(\ell)} \equiv e_{i-1}^{(\ell)}\bar{h} \mod A_i^{(\ell+1)}$$

and

$$(h, a) : \big(e_{i-1}^{(\ell)} + A_i^{(\ell+1)}\big) \mapsto \big(\bar{h}e_{i-1}^{(\ell)}a + A_i^{(\ell+1)}\big),$$

defines an $(H_{k-2\ell}, H_i^{(\ell)})$–bimodule structure on $R_i^{(\ell)}$. Next, the map

$$\begin{aligned} A_i^{(\lambda,\ell)} &\to A_{i-2\ell}^{(\lambda,0)} \otimes_{H_{i-2\ell}} R_i^{(\ell)} \\ \big(x_{(\lambda,\ell)}^{(i)} + A_i^{\rhd(\lambda,\ell)}\big)a &\mapsto \big(c_\lambda^{(i-2\ell)} + H_{i-2\ell}^{\rhd\lambda}\big) \otimes \big(e_{i-1}^{(\ell)} + A_i^{(\ell+1)}\big)a \qquad \text{for } a \in A_i, \end{aligned} \tag{2.7}$$

is well defined. If

$$\big(c_\lambda^{(i-2\ell)} + H_{i-2\ell}^{\rhd\lambda}\big) \otimes \big(e_{i-1}^{(\ell)} + A_i^{(\ell+1)}\big)a = 0,$$

then either $a \in A_i^{(\ell+1)} \subseteq A_i^{\rhd(\lambda,\ell)}$ or there exist $h \in H_{i-2\ell-1}$ and $a' \in A_i$, such that $a = \bar{h}a'$ and $c_\lambda^{(i-2\ell)}h \in H_{i-2\ell}^{\rhd\lambda}$. Thus

$$\big(x_{(\lambda,\ell)}^{(i)} + A_i^{\rhd(\lambda,\ell)}\big)a = \big(\bar{c}_\lambda^{(i)}\bar{h}e_{i-1}^{(\ell)}a' + A_i^{\rhd(\lambda,\ell)}\big) = 0$$

and the map (3.8) is injective. □

Proposition 2.4. *If $(\mu, m-1) \in \hat{A}_{k-2}$, then the map*

$$\mathrm{Ind}_{A_{k-2}}^{A_{k-1}}\big(A_{k-2}^{(\mu,m-1)}\big) = A_{k-2}^{(\mu,m-1)} \otimes_{A_{k-2}} A_{k-1} \to A_k^{(\mu,m)}$$

given, for $a \in A_{k-1}$, by

$$\big(x_{(\mu,m-1)}^{(k-2)} + A_{k-2}^{\rhd(\mu,m-1)}\big) \otimes a \mapsto x_{(\mu,m-1)}^{(k-2)}e_{k-1}a + A_k^{\rhd(\mu,m)} = x_{(\mu,m)}^{(k)}a + A_k^{\rhd(\mu,m)},$$

defines an isomorphism of right A_{k-1}–modules.

Proof. The statement follows from the assumptions that e_{k-1} commutes with A_{k-2} and the map $A_{k-1} \to e_{k-1}A_{k-1}$ given by $a \mapsto e_{k-1}a$ is injective. □

3. Induction and Restriction

In this section, we continue to assume that the pair (2.4) satisfy the hypotheses (A)–(H).

Proposition 3.1. *Let $(\mu, m) \in \hat{A}_{k-1}$, then*

$$A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} A_k^{(m+1)} = A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} e_{k-1}^{(m+1)}A_k \subseteq A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} A_k \tag{3.1}$$

is an inclusion of right A_k–modules such that

$$\big(A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} A_k\big)/\big(A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} e_{k-1}^{(m+1)}A_k\big) \cong A_{k-1}^{(\mu,m)} \otimes_{H_{k-1}^{(m)}} H_k^{(m)}, \tag{3.2}$$

where

$$A_{k-1}^{(\mu,m)} \otimes_{H_{k-1}^{(m)}} H_k^{(m)} \cong \mathrm{Ind}_{H_{k-2m-1}}^{H_{k-2m}}\big(H_{k-2m-1}^{\mu}\big) \otimes_{H_{k-2m}} R_k^{(m)}, \tag{3.3}$$

and

$$A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} e_{k-1}^{(m+1)}A_k \cong \mathrm{Res}_{H_{k-2m-2}}^{H_{k-2m-1}}\big(H_{k-2m-1}^{\mu}\big) \otimes_{H_{k-2m-2}} R_k^{(m+1)}. \tag{3.4}$$

Proof. The inclusion (3.1) follows from the assumption that

$$A_k^{(m+1)} = A_k e_{k-1}^{(m+1)} A_k = A_{k-1} e_{k-1}^{(m+1)} A_k.$$

We prove (3.2). The map

$$\begin{aligned} A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} A_k &\twoheadrightarrow A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} H_k^{(m)}, \\ \big(x_{(\mu,m)}^{(k-1)} + A_{k-1}^{\rhd(\mu,m)}\big) \otimes a &\mapsto \big(x_{(\mu,m)}^{(k-1)} + A_{k-1}^{\rhd(\mu,m)}\big) \otimes \big(a + A_k^{(m+1)}\big), \qquad \text{for } a \in A_k, \end{aligned}$$

is an homomorphism of right A_k–modules. Since $A_{k-1}^{(\mu,m)} A_{k-1}^{(m+1)} = 0$ and $A_{k-1}^{(m+1)} H_k^{(m)} = 0$, it follows that $A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} H_k^{(m)} \cong A_{k-1}^{(\mu,m)} \otimes_{H_{k-1}^{(m)}} H_k^{(m)}$ as right A_k–modules.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] \\ & H_{i-1}^{\mu} \otimes_{A_{i-1}} e_{i-1} A_i \arrow[r] \arrow[d] & H_{i-1}^{\mu} \otimes_{A_{i-1}} e_{i-1} A_i \otimes_{A_i} R_{i+2m}^{(m)} \arrow[d] \\ & H_{i-1}^{\mu} \otimes_{A_{i-1}} A_i \arrow[r] \arrow[d] & H_{i-1}^{\mu} \otimes_{A_{i-1}} A_i \otimes_{A_i} R_{i+2m}^{(m)} \arrow[d] \\ & H_{i-1}^{\mu} \otimes_{H_{i-1}} H_i \arrow[r] \arrow[d] & H_{i-1}^{\mu} \otimes_{H_{i-1}} H_i \otimes_{H_i} R_{i+2m}^{(m)} \arrow[d] \\ & 0 & 0 \end{tikzcd} \tag{3.5}$$

The proof of (3.3) is given in the following steps. Step 1. Since $x_{(\mu,m)}^{(k-1)} = \bar{c}_{\mu}^{(k-2m-1)} e_{k-2}^{(m)}$, the relation $e_{k-2}^{(m)} e_{k-1}^{(m)} e_{k-2}^{(m)} = e_{k-2}^{(m)}$ implies that

$$\big(x_{(\mu,m)}^{(k-1)} + A_{k-1}^{\rhd(\mu,m)}\big) \otimes \big(e_{k-1}^{(m)} + A_{k+1}^{(m+1)}\big) e_{k-2}^{(m)} = \big(x_{(\mu,m)}^{(k-1)} + A_{k-1}^{\rhd(\mu,m)}\big) \otimes \big(1 + A_{k+1}^{(m+1)}\big)$$

as elements of $A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} H_k^{(m)}$. Thus

$$A_{k-1}^{(\mu,m)} \otimes_{H_{k-1}^{(m)}} H_k^{(m)} = A_{k-1}^{(\mu,m)} \otimes R_k^{(m)}$$

as right $H_k^{(m)}$–modules.

Step 2. The assumption that $A_i e_{i-1} = A_{i-1} e_{i-1}$ and $e_{i-1} A_i = e_{i-1} A_{i-1}$ for $i = 1, \dots, k$, implies that $e_{k-2}^{(m)} A_{k-1} e_{k-1}^{(m)} = e_{k-2}^{(m)} A_{k-2m} e_{k-1}^{(m)}$ and

$$A_{k-1}^{(\mu,m)} \otimes \big(e_{k-1}^{(m)} + A_k^{(m+1)}\big) = \big(x_{(\mu,m)}^{(k-1)} + A_{k-1}^{\rhd(\mu,m)}\big) A_{k-2m} \otimes \big(e_{k-1}^{(m)} + A_k^{(m+1)}\big)$$

so

$$A_{k-1}^{(\mu,m)} \otimes R_k^{(m)} = H_{k-2m-1}^{\mu} \otimes_{A_{k-2m-1}} A_{k-2m} \otimes_{A_{k-2m}} R_k^{(m)}.$$

Step 3. We have an inclusion of A_{k-2m}–modules

$$H_{k-2m-1}^{\mu} \otimes_{A_{k-2m-1}} e_{k-2m-1} A_{k-2m} \subseteq H_{k-2m-1}^{\mu} \otimes_{A_{k-2m-1}} A_{k-2m}.$$

Since $A_{k-2m} e_{k-2m-1} A_{k-2m} = A_{k-2m-1} e_{k-2m-1} A_{k-2m}$

$$\begin{aligned} &\big(H_{k-2m-1}^{\mu} \otimes_{A_{k-2m-1}} A_{k-2m}\big) / \big(H_{k-2m-1}^{\mu} \otimes_{A_{k-2m-1}} e_{k-2m-1} A_{k-2m}\big) \\ &\qquad\qquad\qquad \cong H_{k-2m-1}^{\mu} \otimes_{H_{k-2m-1}} H_{k-2m} \end{aligned}$$

via the map

$$\big(\bar{c}_\mu^{(k-2m-1)} + A_{k-2m-1}^{\rhd(\mu,0)}\big) \otimes a \mapsto \big(\bar{c}_\mu^{(k-2m-1)} + A_{k-2m-1}^{\rhd(\mu,0)}\big) \otimes \big(a + A_{k-2m}^{(1)}\big), \qquad \text{for } a \in A_{k-2m}.$$

As

$$H_{k-2m-1}^{\mu} \otimes_{A_{k-2m-1}} e_{k-2m-1}A_{k-2m} \otimes_{A_{k-2m}} R_k^{(m)} = \{0\}.$$

it follows from the diagram (3.5) with $i = k-2m$, that

$$\begin{aligned} H_{k-2m-1}^{\mu} \otimes_{A_{k-2m-1}} A_{k-2m} \otimes_{A_{k-2m}} R_k^{(m)} &= H_{k-2m-1}^{\mu} \otimes_{H_{k-2m-1}} H_{k-2m} \otimes_{H_{k-2m}} R_k^{(m)} \\ &= \mathrm{Ind}_{H_{k-2m-1}}^{H_{k-2m}} \big(H_{k-2m-1}^{\mu}\big) \otimes_{H_{k-2m}} \otimes R_k^{(m)}. \end{aligned}$$

We prove (3.4). The assumption that $A_i e_{i-1} = A_{i-1}e_{i-1}$ and $e_{i-1}A_i = e_{i-1}A_{i-1}$ for $i = 1, \dots, k$, implies that $e_{k-2}^{(m)} A_{k-1} e_{k-1}^{(m+1)} = A_{k-2m-1} e_{k-2}^{(m)} e_{k-1}^{(m+1)}$. Thus

$$\begin{aligned} A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} e_{k-1}^{(m+1)} A_k &= \big(\bar{c}_\mu^{(k-2m-1)} e_{k-2}^{(m)} + A_{k-1}^{\rhd(\mu,m)}\big) A_{k-1} \otimes_{A_{k-1}} e_{k-1}^{(m+1)} A_k \\ &= \big(\bar{c}_\mu^{(k-2m-1)} e_{k-2}^{(m)} + A_{k-1}^{\rhd(\mu,m)}\big) A_{k-2m-1} \otimes_{A_{k-1}} e_{k-1}^{(m+1)} A_k \\ &\cong H_{k-2m-1}^{\mu} \otimes_{A_{k-2m-1}} e_{k-1}^{(m+1)} A_k \end{aligned}$$

as right A_k–modules. Note that

$$A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} A_k^{(m+2)} = A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} e_{k-1}^{(m+2)} A_k = \{0\}. \tag{3.6}$$

Since

$$R_k^{(m+1)} = \big\{ e_{k-1}^{(m+1)} a + A_k^{(m+2)} \mid a \in A_k \big\} \subseteq A_k / A_k^{(m+2)}$$

is an (H_{k-2m-2}, A_k)–bimodule, and A_{k-2m-1} acts on H_{k-2m-1}^{μ} on the right via the map $A_{k-2m-1} \twoheadrightarrow H_{k-2m-1}$, it follows that the diagram

$$\begin{tikzcd} A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} e_{k-1}^{(m+1)} A_k \arrow[r] \arrow[dr] & H_{k-2m-1}^{\mu} \otimes_{H_{k-2m-1}} e_{k-1}^{(m+1)} A_k \arrow[d] \\ & H_{k-2m-1}^{\mu} \otimes_{H_{k-2m-2}} R_k^{(m+1)} \end{tikzcd}$$

commutes and there is an isomorphism

$$\begin{aligned} A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} e_{k-1}^{(m+1)} A_k &\cong H_{k-2m-1}^{\mu} \otimes_{H_{k-2m-2}} R_k^{(m+1)} \\ &= \mathrm{Res}_{H_{k-2m-2}}^{H_{k-2m-1}} \big(H_{k-2m-1}^{\mu}\big) \otimes_{H_{k-2m-2}} R_k^{(m+1)} \end{aligned}$$

of right A_k–modules. $\square$

We establish the following notation.

(1) If $\lambda \in \hat{H}_i$, then we have an H_{i+1} cell–module composition series

$$\{0\} = N_0 \overset{\mu^{(1)}}{\subseteq} N_1 \overset{\mu^{(2)}}{\subseteq} \cdots \overset{\mu^{(p)}}{\subseteq} N_p = \mathrm{Ind}_{H_i}^{H_{i+1}} \big(H_i^{\lambda}\big),$$

where $\mu^{(s)} \rhd \mu^{(t)}$ whenever $t > s$. For $j = 1, \dots, p$, fix $u_{\mu \to \lambda^{(j)}}^{(i+1)} \in H_{i+1}$, such that if

$$N_j = \sum_{s \leqslant j} \big(c_\lambda^{(i)} + H_i^{\rhd\lambda}\big) \otimes u_{\lambda \to \mu^{(s)}}^{(i+1)} H_{i+1},$$

then

$$\big(c_{\mu^{(j)}}^{(i+1)} + H_{i+1}^{\rhd\mu^{(j)}}\big) \mapsto \big(c_\lambda^{(i)} + H_i^{\rhd\lambda}\big) \otimes u_{\lambda \to \mu^{(j)}}^{(i+1)} + N_{j-1}$$

under the isomorphism $H_{i+1}^{\mu^{(j)}} \cong N_j / N_{j-1}$.

(2) If $\mu \in \hat{H}_{i+1}$, then we have an H_i cell–module composition series

$$\{0\} = M_0 \overset{\lambda^{(1)}}{\subseteq} M_1 \overset{\lambda^{(2)}}{\subseteq} \cdots \overset{\lambda^{(r)}}{\subseteq} M_r = \operatorname{Res}^{H_{i+1}}_{H_i}\left(H^{\mu}_{i+1}\right)$$

where $\lambda^{(s)} \rhd \lambda^{(t)}$ whenever $t > s$. For $j = 1, \dots, r$, fix $d^{(i+1)}_{\lambda^{(j)}\to\mu} \in H_{i+1}$, such that if

$$M_j = \sum_{s \leqslant j} \left(c^{(i+1)}_{\mu} + H^{\rhd\lambda}_{i+1}\right) d^{(i+1)}_{\lambda^{(s)}\to\mu} H_i,$$

then

$$\left(c^{(i)}_{\lambda^{(j)}} + H_i^{\rhd\mu^{(j)}}\right) \mapsto \left(c^{(i+1)}_{\mu} + H^{\rhd\mu}_{i+1}\right) d^{(i+1)}_{\lambda^{(j)}\to\mu} + M_{j-1}$$

under the isomorphism $H_i^{\lambda^{(j)}} \cong M_j/M_{j-1}$.

In the above setting, fix

$$\bar{u}^{(i+1)}_{\mu\to\lambda^{(j)}} \in A_{i+1} \qquad \text{such that} \qquad u^{(i+1)}_{\mu\to\lambda^{(j)}} = \bar{u}^{(i+1)}_{\mu\to\lambda^{(j)}} + A^{(1)}_i \qquad \text{for } j = 1, \dots, p.$$

Similarly, let

$$\bar{d}^{(i+1)}_{\mu\to\lambda^{(j)}} \in A_{i+1} \qquad \text{such that} \qquad d^{(i+1)}_{\lambda^{(j)}\to\mu} = \bar{d}^{(i+1)}_{\lambda^{(j)}\to\mu} + A^{(1)}_i \qquad \text{for } j = 1, \dots, r.$$

In the next statement we make explicit the A_k–module filtrations given in Proposition 3.1.

Theorem 3.2. *Let $(\lambda, \ell) \in \hat{A}_{k-1}$ and*

$$\left\{(\mu^{(j)}, \ell+1), (\mu^{(j+r)}, \ell) \mid i = 1, \dots, r \text{ and } j = 1, \dots, p\right\}$$

be an indexing of the set

$$\{(\mu, m) \mid (\mu, m) \in \hat{A}_k \text{ and } (\lambda, \ell) \to (\mu, m)\}$$

such that $(\mu^{(s)}, m_s) \rhd (\mu^{(t)}, m_t)$ whenever $t > s$. For $j = 1, \dots, r$, let $U_j \subseteq \operatorname{Ind}^{A_k}_{A_{k-1}}\left(A^{(\lambda,\ell)}_{k-1}\right)$ be the A_k–submodule

$$U_j = \sum_{\substack{s \leqslant j \\ (\lambda,\ell)\to(\mu^{(s)},\ell+1)}} \left(x^{(k-1)}_{(\lambda,\ell)} + A^{\rhd(\lambda,\ell)}_{k-1}\right) \otimes \bar{d}^{(k-2\ell-1)}_{\lambda\to\mu^{(s)}} e^{(\ell+1)}_{k-1} A_k.$$

Define also, for $j = 1, \dots, p$, the A_k–submodule $U_{r+j} \subseteq \operatorname{Ind}^{A_k}_{A_{k-1}}\left(A^{(\lambda,\ell)}_{k-1}\right)$ by

$$U_{j+r} = U_r + \sum_{\substack{s \leqslant j \\ (\lambda,\ell)\to(\mu^{(r+s)},\ell)}} \left(x^{(k-1)}_{(\lambda,\ell)} + A^{\rhd(\lambda,\ell)}_{k-1}\right) \otimes e^{(\ell)}_{k-1} \bar{u}^{(k-2\ell)}_{\lambda\to\mu^{(r+s)}} A_k.$$

Then

$$\{0\} = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{r+p} = \operatorname{Ind}^{A_k}_{A_{k-1}}\left(A^{(\lambda,\ell)}_{k-1}\right)$$

is a filtration by A_{k+1}–submodules, with multiplicity free subquotients

$$A^{(\mu^{(j)}, m_j)}_{k+1} \cong U_j/U_{j-1}, \qquad \text{for } j = 1, \dots, r+p, \text{ where} \qquad m_j = \begin{cases} \ell+1, & \text{if } j \leqslant r, \\ \ell, & \text{if } r < j. \end{cases}$$

Proof. By Proposition 3.1 there is an inclusion of A_k–modules

$$A^{(\lambda,\ell)}_{k-1} \otimes_{A_{k-1}} e^{(\ell+1)}_{k-1} A_k \subseteq A^{(\lambda,\ell)}_{k-1} \otimes_{A_{k-1}} A_k$$

such that

$$\begin{aligned} A^{(\lambda,\ell)}_{k-1} \otimes_{A_{k-1}} e^{(\ell+1)}_{k-1} A_k &= \left(x^{(k-2\ell-1)}_{(\lambda,\ell)} + A^{\rhd(\lambda,\ell)}_{k-1}\right) \otimes_{A_{k-2\ell-2}} R^{(\ell+1)}_k \\ &\cong \operatorname{Res}^{H_{k-2\ell-1}}_{H_{k-2\ell-2}}\left(H^{\lambda}_{k-2\ell-1}\right) \otimes_{H_{k-2\ell-2}} R^{(\ell+1)}_k, \end{aligned} \tag{3.7}$$

Define an R–module homomorphism $\varphi_{\ell,k-1} : A_{k-1} \to A_{k-2\ell-1}$ by

$$e_{k-2}^{(\ell)} a e_{k-1}^{(\ell+1)} = \varphi_{\ell,k-1}(a) e_{k-2}^{(\ell)} e_{k-1}^{(\ell+1)} \qquad \text{for } a \in A_{k-1},$$

and note that $\varphi_{\ell,k-1}(a) = a$ for all $a \in A_{k-2\ell-1}$. In light of (3.6), the isomorphism (3.7) is realised by the A_k–module homomorphism which maps

$$x_{k-1}^{(\lambda,\ell)} a \otimes e_{k-1}^{(\ell+1)} \mapsto \big(c_\lambda^{(k-2\ell-1)} + H_{k-2\ell-1}^{\rhd\lambda}\big)\big(\varphi_{\ell,k-1}(a) + A_{k-2\ell-1}^{(1)}\big) \otimes \big(e_{k-1}^{(\ell+1)} + A_k^{(\ell+2)}\big), \tag{3.8}$$

for $a \in A_{k-1}$. Since the set

$$\Big\{\big(c_\lambda^{(k-2\ell-1)} + H_{k-2\ell-1}^{\rhd\lambda}\big) d_{\mu\to\lambda}^{(k-2\ell-1)} \,\big|\, \mu \in \hat{H}_{k-2\ell-2},\, \mu \to \lambda\Big\}$$

generates $\mathrm{Res}_{A_{k-2\ell-2}}^{A_{k-2\ell-1}}\big(H_{k-2\ell-1}^{\lambda}\big)$ as an $A_{k-2\ell-2}$–module, using the isomorphism (3.8) and the assumptions on the structure of $\mathrm{Res}_{A_{k-2\ell-2}}^{A_{k-2\ell-1}}\big(H_{k-2\ell-1}^{\lambda}\big)$, it follows that

$$\{0\} = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_r = A_{k-1}^{(\lambda,\ell)} \otimes_{A_{k-1}} e_{k-1}^{(\ell+1)} A_k$$

is a filtration by A_k–submodules, with

$$U_j/U_{j-1} \cong H_{k-2\ell-2}^{\mu^{(j)}} \otimes_{H_{k-2\ell-2}} R_k^{(\ell+1)} \cong A_k^{(\mu^{(j)},\ell+1)} \qquad \text{for } j = 1, \dots, r.$$

Proposition 3.1 has shown that

$$\begin{aligned}\big(A_{k-1}^{(\lambda,\ell)} \otimes_{A_{k-1}} A_k\big)/U_r &= \big(x_{(\lambda,\ell)}^{(k-1)} + A_{k-1}^{\rhd(\lambda,\ell)}\big) A_{k-2\ell} \otimes_{A_{k-2\ell}} R_k^{(\ell)} \\ &\cong \mathrm{Ind}_{H_{k-2\ell-1}}^{H_{k-2\ell}}\big(H_{k-2\ell-1}^{\lambda}\big) \otimes_{H_{k-2\ell}} R_k^{(\ell)}.\end{aligned} \tag{3.9}$$

Define an R–module homomorphism $\vartheta_{\ell,k-1} : A_{k-1} \to A_{k-2\ell+1}$ by

$$e_{k-2}^{(\ell)} a e_{k-1}^{(\ell)} = e_{k-2}^{(\ell)} e_{k-1}^{(\ell)} \vartheta_{\ell,k-1}(a) \qquad \text{for } a \in A_{k-1},$$

and note that $\vartheta_{\ell,k-1}(a) = a$ for all $a \in A_{k-2\ell+1}$. Since

$$A_{k-1}^{(\lambda,\ell)} A_{k-2\ell}^{(1)} \otimes_{A_{k-1}} e_{k-1}^{(\ell)} \subseteq A_{k-1}^{(\lambda,\ell)} \otimes e_{k-1}^{(\ell+1)} A_k = U_r,$$

the isomorphism (3.9) is realised by the A_k–module map

$$\begin{aligned}&\big(x_{(\lambda,\ell)}^{(k-1)} + A_{k-1}^{\rhd(\lambda,\ell)}\big) a \otimes \big(e_{k-1}^{(\ell)} + A_k^{(\ell+1)}\big) \mapsto \\ &\qquad \big(c_\lambda^{(k-2\ell-1)} + H_{k-2\ell-1}^{\rhd\lambda}\big) \otimes \big(\vartheta_{\ell,k-1}(a) + A_{k-2\ell}^{(1)}\big) \otimes \big(e_{k-1}^{(\ell)} + A_k^{(\ell+1)}\big),\end{aligned} \tag{3.10}$$

for $a \in A_{k-1}$. Since the set

$$\Big\{\big(c_\lambda^{(k-2\ell-1)} + H_{k-2\ell-1}^{\rhd\lambda}\big) \otimes u_{\lambda\to\mu}^{(k-2\ell)} \,\big|\, \mu \in \hat{H}_{k-2\ell},\, \lambda \to \mu\Big\}$$

generates $\mathrm{Ind}_{A_{k-2\ell-1}}^{A_{k-2\ell}}\big(H_{k-2\ell-1}^{\lambda}\big)$ as an $A_{k-2\ell}$–module, using the map (3.10) and the assumptions on the structure of $\mathrm{Ind}_{A_{k-2\ell-1}}^{A_{k-2\ell}}\big(H_{k-2\ell-1}^{\lambda}\big)$, it follows that

$$U_r \subseteq U_{r+1} \subseteq U_{r+2} \subseteq \cdots \subseteq U_{r+p} = A_{k-1}^{(\lambda,\ell)} \otimes_{A_{k-1}} A_k$$

is a filtration by A_k–submodules, such that

$$U_j/U_{j-1} \cong H_{k-2\ell-2}^{\mu^{(j)}} \otimes_{H_{k-2\ell-2}} R_k^{(\ell+1)} \cong A_k^{(\mu^{(j)},m)} \qquad \text{for } j = r+1, \dots, r+p.$$

This completes the proof of the theorem. □

If $(\mu, m) \in \hat{A}_{k-2}$, then next statement gives an explicit filtration by A_k–modules of the A_{k+1}–module $A_{k+1}^{(\mu,m+1)} \cong A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} A_k$.

Theorem 3.3. *Let* $(\mu, m+1) \in \hat{A}_{k+1}$ *and*

$$\Big\{(\lambda^{(i)}, m+1), (\lambda^{(r+j)}, m) \mid i = 1, \dots, r \text{ and } j = 1, \dots, p\Big\}$$

be an indexing of the set

$$\big\{(\lambda, \ell) \in \hat{A}_k \mid (\lambda, \ell) \to (\mu, m+1)\big\}$$

such that $(\lambda^{(s)}, \ell_s) \rhd (\lambda^{(t)}, \ell_t)$ *whenever* $t > s$. *For* $j = 1, \dots, r$, *let* $U_j \subseteq A_{k+1}^{(\mu,m+1)}$ *be the* A_k*–submodule*

$$U_j = \sum_{\substack{s \leqslant j \\ (\lambda^{(s)}, m+1) \to (\mu, m+1)}} \big(x_{(\mu,m+1)}^{(k+1)} \bar{d}_{\lambda^{(s)} \to \mu}^{(k-2m-1)} e_{k-1}^{(m+1)} + A_{k+1}^{\rhd(\mu,m+1)}\big) A_k.$$

Define also, for $j = 1, \dots, p$, *the* A_k*–submodule* $U_{j+r} \subseteq A_{k+1}^{(\mu,m+1)}$ *by*

$$U_{r+j} = U_r + \sum_{\substack{s \leqslant j \\ (\lambda^{(r+s)}, m) \to (\mu, m+1)}} \big(x_{(\mu,m+1)}^{(k+1)} e_{k-1}^{(m)} \bar{u}_{\mu \to \lambda^{(r+s)}}^{(k-2m)} + A_{k+1}^{\rhd(\mu,m+1)}\big) A_k.$$

Then

$$\{0\} = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{r+p} = A_{k+1}^{(\mu,m+1)}$$

is a filtration by A_k*–modules with multiplicity free subquotients*

$$A_k^{(\lambda^{(j)}, \ell_j)} \cong U_j / U_{j-1} \qquad \text{for } j = 1, \dots, r+p, \text{ where} \qquad \ell_j = \begin{cases} m+1, & \text{if } j \leqslant r, \\ m, & \text{if } r < j. \end{cases}$$

Proof. Let $(\mu, m) \in \hat{A}_{k-1}$. We make the identification

$$A_{k+1}^{(\mu,m+1)} = A_{k-1}^{(\mu,m)} \otimes_{A_{k-1}} A_k$$

via the map

$$\big(x_{(\mu,m)}^{(k-1)} + A_{k-1}^{\rhd(\mu,m)}\big) \otimes a \mapsto \big(x_{(\mu,m+1)}^{(k+1)} + A_{k+1}^{\rhd(\mu,m+1)}\big) a \qquad \text{for } a \in A_k,$$

and proceed as in Theorem 3.2. □

4. The Murphy Basis

In this section we continue to assume that the pair (2.4) satisfy the hypotheses (A)–(H) and use Theorem 3.3 to construct an explicit Murphy–type cellular basis for each algebra in the tower $R = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$.

If $(\lambda, \ell) \in \hat{A}_k$, $(\mu, m) \in \hat{A}_{k+1}$, and $\lambda \to \mu$ in $\hat{H}$, define

$$a_{(\lambda,\ell)\to(\mu,m)}^{(k+1)} = \begin{cases} \bar{d}_{\lambda\to\mu}^{(k-2m+1)} e_{k-1}^{(m)}, & \text{if } \ell = m, \\ e_{k-1}^{(m-1)} \bar{u}_{\mu\to\lambda}^{(k-2m+2)}, & \text{if } \ell = m-1. \end{cases} \tag{4.1}$$

For $\mathfrak{t} = ((\lambda^{(0)}, \ell_0), (\lambda^{(1)}, \ell_1), \dots, (\lambda^{(k)}, \ell_k)) \in \hat{A}_k^{(\lambda,\ell)}$, let

$$a_{\mathfrak{t}}^{(k)} = a_{(\lambda^{(k-1)}, \ell_{k-1}) \to (\lambda^{(k)}, \ell_k)}^{(k)} a_{(\lambda^{(k-2)}, \ell_{k-2}) \to (\lambda^{(k-1)}, \ell_{k-1})}^{(k-1)} \cdots a_{(\lambda^{(0)}, \ell_0) \to (\lambda^{(1)}, \ell_1)}^{(1)}. \tag{4.2}$$

In the next theorem, we have used the abbreviation $a_{\mathfrak{t}} = a_{\mathfrak{t}}^{(k)}$ for $\mathfrak{t} \in \hat{A}_k^{(\lambda,\ell)}$ and $(\lambda, \ell) \in \hat{A}_k$.

Theorem 4.1. *Let* $R = A_0 \subseteq A_1 \subseteq \cdots$, *and* $R = H_0 \subseteq H_1 \subseteq \cdots$, *be algebras satisfying assumptions* (A)–(H) *above. If* $k = 1, 2, \dots$, *then the set*

$$\mathscr{A}_k = \Big\{a_{\mathfrak{s}}^* x_{(\lambda,\ell)}^{(k)} a_{\mathfrak{t}} \,\Big|\, \mathfrak{s}, \mathfrak{t} \in \hat{A}_k^{(\lambda,\ell)},\ (\lambda, \ell) \in \hat{A}_k,\ \text{and } \ell = 0, 1, \dots \lfloor k/2 \rfloor\Big\} \tag{4.3}$$

is an R*–basis for* A_k. *Moreover, the following statements hold:*

(1) Given $(\lambda,\ell)\in\hat{A}_k$, $\mathfrak{t}\in\hat{A}_k^{(\lambda,\ell)}$, and $a\in A_k$, there exist $r_\mathfrak{u}\in R$, for $\mathfrak{u}\in\hat{A}_k^{(\lambda,\ell)}$, such that for all $\mathfrak{s}\in\hat{A}_k^{(\lambda,\ell)}$,

$$a_\mathfrak{s}^* x_{(\lambda,\ell)}^{(k)} a_\mathfrak{t} a \equiv \sum_{\mathfrak{u}\in\hat{A}_k^{(\lambda,\ell)}} r_\mathfrak{u} a_\mathfrak{s}^* x_{(\lambda,\ell)}^{(k)} a_\mathfrak{u} \mod A_k^{\rhd(\lambda,\ell)}, \tag{4.4}$$

where $A_k^{\rhd(\lambda,\ell)}$ is freely generated as an R–module by

$$\left\{ a_\mathfrak{s}^* x_{(\mu,m)}^{(k)} a_\mathfrak{t} \,\middle|\, \mathfrak{s},\mathfrak{t}\in\hat{A}_k^{(\mu,m)} \text{ where } (\mu,m)\in\hat{A}_k \text{ and } (\mu,m)\rhd(\lambda,\ell) \right\}. \tag{4.5}$$

(2) If $(\lambda,\ell)\in\hat{A}_k$, and $\mathfrak{s},\mathfrak{t}\in\hat{A}_k^{(\lambda,\ell)}$, then $*: a_\mathfrak{s}^* x_{(\lambda,\ell)}^{(k)} a_\mathfrak{t} \mapsto a_\mathfrak{t}^* x_{(\lambda,\ell)}^{(k)} a_\mathfrak{s} \mod A_k^{\rhd(\lambda,\ell)}$.

Proof. We first observe that the statement (2) of the theorem follows from the definition of the ideal $A_k^{\rhd(\lambda,\ell)}$ in (2.5) and the assumptions that $*: e_{k-1}^{(\ell)} \to e_{k-1}^{(\ell)}$ and $*: c_\lambda^{(k-2\ell)} \to c_\lambda^{(k-2\ell)} \mod H_{k-2\ell}^{\rhd\lambda}$.

We show that the set $\mathscr{A}_k$ in (4.3) is an R–basis for A_k. If $t=\lfloor k/2\rfloor$, then

$$\{0\}\subseteq A_k^{(t)}\subseteq A_k^{(t-1)}\subseteq\cdots\subseteq A_k^{(0)}=A_k$$

is a filtration by two–sided ideals of A_k and, as an R–module,

$$A_k=\bigoplus_{\ell=0}^{\lfloor k/2\rfloor} H_k^{(\ell)}.$$

By the assumptions on the tower $R=H_0\subseteq H_1\subseteq\cdots$, it follows that

$$\left\{ a_\mathfrak{s}^* x_{(\lambda,0)}^{(k)} a_\mathfrak{t} + A_k e_{k-1} A_k \,\middle|\, \mathfrak{s},\mathfrak{t}\in\hat{A}_k^{(\lambda,0)},\ \lambda\in\hat{H}_k \right\}$$

is an R–basis for $A_k/(A_k e_{k-1} A_k)\cong H_k$. Let $m>0$ and $(\mu,m)\in\hat{A}_k$ and take an indexing

$$\{(\lambda^{(i)},m),(\lambda^{(r+j)},m-1)\,|\, i=1,\dots,r \text{ and } j=1,\dots,p\}$$

of the set

$$\{(\lambda,\ell)\in\hat{A}_{k-1}\mid(\lambda,\ell)\to(\mu,m)\}$$

such that $(\lambda^{(s)},\ell_s)\rhd(\lambda^{(t)},\ell_t)$ whenever $t>s$. Let

$$U_j=\big(x_{(\mu,m)}^{(k)}+A_k^{\rhd(\mu,m)}\big)a_{(\lambda^{(j)},\ell_j)\to(\mu,m)}A_{k-1}$$

so that

$$\{0\}=U_0\subseteq U_1\subseteq\cdots\subseteq U_{r+p}=A_k^{(\mu,m)}$$

is a filtration of $A_k^{(\mu,m)}$ by A_{k-1}–modules,where

$$\left(x_{(\lambda^{(j)},\ell_j)}^{(k-1)}+A_{k-1}^{\rhd(\lambda^{(j)},\ell_j)}\right)\mapsto\left(x_{(\mu,m)}^{(k)}+A_k^{\rhd(\mu,m)}\right)a_{(\lambda^{(j)},\ell_j)\to(\mu,m)}+U_{j-1}$$

under the isomorphism

$$U_j/U_{j-1}\cong A_{k-1}^{(\lambda^{(j)},\ell_j)}, \qquad \text{for } j=1,\dots,r+p. \tag{4.6}$$

Using the isomorphism (4.6) to pull bases for $A_{k-1}^{(\lambda^{(j)},\ell_j)}$ back onto bases for U_j/U_{j-1}, it follows by induction on k that

$$\left\{ x_{(\mu,m)}^{(k)} a_\mathfrak{t} + A_k^{\rhd(\mu,m)} \,\middle|\, \mathfrak{t}\in\hat{A}_k^{(\mu,m)} \right\}$$

is an R–basis for

$$A_k^{(\mu,m)}=\left\{ x_{(\mu,m)}^{(k)} a + A_k^{\rhd(\mu,m)} \,\middle|\, a\in\hat{A}_k \right\}\subseteq A_k/A_k^{\rhd(\mu,m)}.$$

Thus, using the anti–involution $*: A_k \to A_k$ and the statement (2) of the theorem, it follows that

$$\left\{a_{\mathfrak{s}}^* x_{(\mu,m)}^{(k)} a_{\mathfrak{t}} + A_k^{\rhd(\mu,m)} \mid \mathfrak{s},\mathfrak{t} \in \hat{A}_k^{(\mu,m)}\right\}$$

is an R–basis for the (A_k, A_k)–bimodule

$$\left\{a x_{(\mu,m)}^{(k)} a' + A_k^{\rhd(\mu,m)} \mid a, a' \in \hat{A}_k\right\} \subseteq A_k/A_k^{\rhd(\mu,m)}.$$

By induction on the order on $\hat{A}_k$ it now follows that

$$\left\{a_{\mathfrak{s}}^* x_{(\lambda,\ell)}^{(k)} a_{\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \hat{A}_k^{(\mu,m)},\ \mu \in \hat{H}_{k-2m}\right\}$$

is an R–basis for $H_k^{(m)}$. Thus the set $\mathscr{A}_k$ in (4.3) is an R–basis for A_k.

The statement (1) of the theorem now follows from the above discussion and the definition of the ideal $A_k^{\rhd(\lambda,\ell)}$, for $(\lambda,\ell) \in \hat{A}_k$, in (2.5). □

The author is is grateful to Arun Ram for pointing out that the inductive procedure used to construct bases in the proof of Theorem 4.1 in fact yields the closed expression for the operators $\{a_{\mathfrak{t}}^{(k)} \mid \mathfrak{t} \in \hat{A}_k^{(\lambda,\ell)}\}$ in (4.2).

Proposition 4.2. *Let $(\mu,m) \in \hat{A}_i$ and $\{(\lambda^{(1)},\ell_1),\dots,(\lambda^{(p)},\ell_p)\}$ be an indexing of the set*

$$\{(\lambda,\ell) \in \hat{A}_{i-1} \mid (\lambda,\ell) \to (\mu,m)\}$$

such that $(\lambda^{(s)},\ell_s) \rhd (\lambda^{(t)},\ell_t)$ whenever $t > s$. For $j = 1,\dots,p$, let

$$M_j = \sum_{\substack{\mathfrak{s}\in\hat{A}_i^{(\mu,m)} \\ \mathrm{Shape}(\mathfrak{s}|_{i-1}) \trianglerighteq (\lambda^{(j)},\ell_j)}} \big(x_{(\mu,m)}^{(i)} + A_i^{\rhd(\mu,m)}\big) a_{\mathfrak{s}}^{(i)} A_{i-1}.$$

Then

$$\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_p = A_i^{(\mu,m)}$$

is a filtration by right A_{i-1}–modules and, for $j = 1,\dots,p$, the R–linear map

$$\begin{aligned} A_{i-1}^{(\lambda^{(j)},\ell_j)} &\longrightarrow M_j/M_{j-1} \\ \big(x_{(\lambda^{(j)},\ell_j)}^{(i-1)} + A_{i-1}^{\rhd(\mu,m)}\big) a_{\mathfrak{u}} &\longmapsto \big(x_{(\mu,m)}^{(i)} + A_i^{\rhd(\mu,m)}\big) a_{\mathfrak{t}} + M_{j-1}, \end{aligned} \tag{4.7}$$

for $\mathfrak{u} \in \hat{A}_{i-1}^{(\lambda^{(j)},\ell_j)}$ and $\mathfrak{t} \in \hat{A}_i^{(\mu,m)}$ such that $\mathfrak{t}|_{i-1} = \mathfrak{u}$, is an isomorphism of right A_{i-1}–modules.

5. Applications

If the pair (2.4) satisfy the assumptions (A)–(H), then by Theorem 4.1, a cellular basis for A_i, for $i \geqslant 0$, of the form (4.3) is determined explicitly by the following data:

(1) The branching diagram $\hat{H}$ whose vertices on level i, for $i \geqslant 0$, consist of the elements of the partially ordered set $\hat{H}_i$.

(2) The maps

$$*: A_i \to A_i \qquad \text{and} \qquad A_i/(A_i e_{i-1} A_i) \xrightarrow{\cong} H_i. \tag{5.1}$$

(3) The branching diagram $\hat{A}$ whose vertices on level i, for $i \geqslant 0$, consist of the elements of the partially ordered set

$$\hat{A}_i = \big\{(\lambda,\ell) \mid \lambda \in \hat{H}_{i-2\ell},\ \text{for } \ell = 0,1,\dots,\lfloor i/2 \rfloor\big\} \tag{5.2}$$

with an edge $(\lambda,\ell) \to (\mu,m)$, for $(\lambda,\ell) \in \hat{A}_{i-1}$ and $(\mu,m) \in \hat{A}_i$, if either

(a) $\ell = m$ and there is an edge $\lambda \to \mu$ from level $i-2m-1$ to level $i-2m$ in $\hat{H}$, or

(b) $\ell = m-1$ and there is an edge $\mu \to \lambda$ from level $i-2m$ to level $i-2m+1$ in $\hat{H}$.

(4) The elements

$$x^{(k)}_{(\mu,m)} = \bar{c}^{(k-2\ell)}_{\mu} e^{(\ell)}_{k-1}, \qquad\qquad \text{for } (\mu,m) \in \hat{A}_k. \tag{5.3}$$

(5) The elements

$$a^{(i+1)}_{(\lambda,\ell)\to(\mu,m)} = \begin{cases} \bar{d}^{(i-2m+1)}_{\lambda\to\mu} e^{(m)}_{i-1}, & \text{if } \ell = m, \\ e^{(m-1)}_{i-1} \bar{u}^{(i-2m+2)}_{\mu\to\lambda}, & \text{if } \ell = m-1, \end{cases} \tag{5.4}$$

for each edge $(\lambda,\ell) \to (\mu,m)$ from level i to level $i+1$ in the branching diagram $\hat{A}$.

In the examples below, we use Theorem 4.1 and the bases for the Iwahori–Hecke algebra of the symmetric group given by Murphy [Mu] to produce explicit cellular bases, in the form of the data (5.1)–(5.4), for important examples of algebras obtained by a Jones basic construction. By Theorems 3.2 and 3.3 our construction will yield cellular bases that are compatible with induction and restriction on cell modules. Note that in each example considered below, the bases obtained are cellular in the strict sense of [GL]. We first establish some notation.

5.1. **Combinatorics.** Let k denote a non–negative integer and $\mathfrak{S}_k$ be the symmetric group acting on $\{1,\dots,k\}$ on the right. For i an integer, $1 \leqslant i < k$, let s_i denote the transposition $(i,i+1)$. Then $\mathfrak{S}_k$ is presented as a Coxeter group by generators $s_1, s_2, \dots, s_{k-1}$, with the relations

$$\begin{aligned} &s_i^2 = 1, && \text{for } i = 1,\dots,k-1, \\ &s_i s_j = s_j s_i, && \text{for } j \neq i+1. \\ &s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, && \text{for } i = 1,\dots,k-2. \end{aligned}$$

An product $w = s_{i_1} s_{i_2} \cdots s_{i_j}$ in which j is minimal is called a *reduced expression* for w and $j = \ell(w)$ is the *length* of w. If $i,j = 1,\dots,k$, define

$$w_{i,j} = \begin{cases} s_i s_{i+1} \cdots s_{j-1}, & \text{if } j \geqslant i, \\ s_{i-1} s_{i-2} \cdots s_j, & \text{if } i > j. \end{cases}$$

If $k > 0$, a *partition* of k is a non–increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of integers, $\lambda_i \geqslant 0$, such that $\sum_{i\geqslant 1} \lambda_i = k$; otherwise, if $k = 0$, write $\lambda = \emptyset$ for the empty partition. The fact that λ is a partition of k will be denoted by $\lambda \vdash k$. If λ is a partition, we will also write $|\lambda| = \sum_{i\geqslant 1} \lambda_i$. The integers $\{\lambda_i \mid \text{for } i \geqslant 1\}$ are the *parts* of λ. If $\lambda \vdash k$, the *Young diagram* of λ is the set

$$[\lambda] = \{(i,j) \mid \lambda_i \geqslant j \geqslant 1 \text{ and } i \geqslant 1\} \subseteq \mathbb{N} \times \mathbb{N}.$$

The elements of $[\lambda]$ are the *nodes* of λ and more generally a node is a pair $(i,j) \in \mathbb{N} \times \mathbb{N}$. The diagram $[\lambda]$ is traditionally represented as an array of boxes with λ_i boxes on the i–th row. For example, if $\lambda = (3,2)$, then $[\lambda] = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}$. Let $[\lambda]$ be the diagram of a partition. Usually, we will identify the partition λ with its diagram and write λ in place of $[\lambda]$.

The dominance $\trianglerighteq$ on partitions of k is defined as follows: if $\lambda \vdash k$ and $\mu \vdash k$, then $\lambda \trianglerighteq \mu$ if

$$\textstyle\sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i \qquad\qquad \text{for all } j \geqslant 1.$$

We write $\lambda \triangleright \mu$ to mean that $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$.

Let $\lambda \vdash k$. A λ–tableau $\mathfrak{t}$ from the nodes of the diagram $[\lambda]$ to the integers $\{1,2,\dots,k\}$. A given λ–tableau $\mathfrak{t} : [\lambda] \to \{1,2,\dots,k\}$ can be represented by labelling the nodes of the diagram $[\lambda]$ with the integers $1,2,\dots,k$. For example, if $k = 6$ and $\lambda = (3,2,1)$,

$$\mathfrak{t} = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 3 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \tag{5.5}$$

represents a λ–tableau. If $\lambda \vdash k$, let $\mathfrak{t}^\lambda$ denote the λ–tableau in which $1, 2, \ldots, k$ are entered in increasing order from left to right along the rows of $[\lambda]$. Thus in the previous example where $k = 6$ and $\lambda = (3, 2, 1)$,

$$\mathfrak{t}^\lambda = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\,. \tag{5.6}$$

The tableau $\mathfrak{t}^\lambda$ is the *row reading tableau* of shape λ. The symmetric group $\mathfrak{S}_k$ acts on the set of λ–tableaux on the right by permuting the integer labels of the nodes of $[\lambda]$. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\,(2,4)(3,6,5) = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 3 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}\,.$$

If $\lambda \vdash k$, the *Young subgroup* $\mathfrak{S}_\lambda$ is defined to be the row stabiliser of $\mathfrak{t}^\lambda$ in $\mathfrak{S}_k$. For instance, when $k = 6$ and $\lambda = (3, 2, 1)$, as in (5.6), then $\mathfrak{S}_\lambda = \langle s_1, s_2, s_4 \rangle$.

5.2. **Iwahori–Hecke algebras of the symmetric group.** Let R be an integral domain and q be a unit in R. Let $\mathcal{H}_k = \mathcal{H}_k(q^2)$ denote the Iwahori–Hecke algebra of the symmetric group which is presented by the generators $T_1, \ldots, T_{k-1}$, and the relations

$$\begin{aligned}
&T_i T_j = T_j T_i, && \text{if } j \neq i+1,\\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, && \text{for } i = 1, \ldots, k-2,\\
&(T_i - q)(T_i + q^{-1}) = 0, && \text{for } i = 1, \ldots, k-1.
\end{aligned}$$

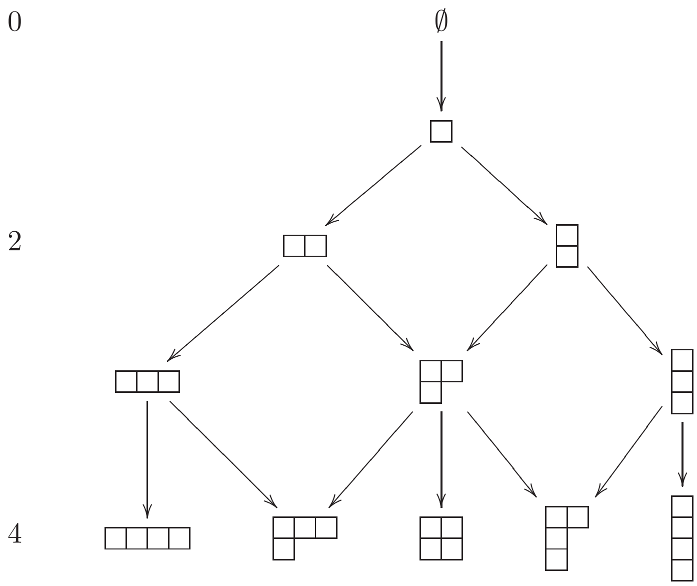
If $v \in \mathfrak{S}_k$, and $v = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is a reduced expression for w in $\mathfrak{S}_k$, then $T_v = T_{i_1} T_{i_2} \cdots T_{i_\ell}$ is well defined in $\mathcal{H}_k(q^2)$ and $\{T_v \mid v \in \mathfrak{S}_k\}$ freely generates $\mathcal{H}_k(q^2)$ as an R–module. The R–module map $* : T_v \mapsto T_{v^{-1}}$ is an algebra anti–automorphism of $\mathcal{H}_k(q^2)$. If $i, j = 1, \ldots, k$, let

$$T_{i,j} = \begin{cases} T_i T_{i+1} \cdots T_{j-1}, & \text{if } j \geqslant i,\\ T_{i-1} T_{i-2} \cdots T_j, & \text{if } i > j. \end{cases}$$

The branching diagram of the tower $R = \mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \cdots$ is the graph $\hat{\mathcal{H}}$ with:

(1) vertices on level i: $\hat{\mathcal{H}}_i = \{\lambda \mid \lambda \vdash i\}$, ordered by the dominance order $\trianglerighteq$ on partitions.
(2) an edge $\lambda \to \mu$, for $\lambda \in \hat{\mathcal{H}}_i$ and $\mu \in \hat{\mathcal{H}}_{i+1}$, if μ is obtained by adding a node to λ.

The first few levels of the graph $\hat{\mathcal{H}}$ are given in (5.7).



(5.7)

The generic branching rules encoded in the graph $\hat{\mathcal{H}}$ can be made explicit using the cellular basis for $\mathcal{H}_k$ given by Murphy [Mu]. The Murphy basis is constructed as follows.

Let $\mu \in \hat{\mathcal{H}}_i$. Define

$$\hat{\mathcal{H}}_i^\mu = \{\mathfrak{t} \mid \mathfrak{t} \text{ is a path from level 0 to the vertex } \mu \text{ in level } i \text{ of } \hat{\mathcal{H}}\}.$$

Given $\lambda \in \hat{\mathcal{H}}_{i-1}$, such that $\mu = \lambda \cup \{(j, \mu_j)\}$, define

$$d_{\lambda\to\mu}^{(i)} = T_{i,a_j}, \qquad \text{where} \qquad a_j = \sum_{r=1}^{j} \mu_r.$$

If $\lambda \in \hat{\mathcal{H}}_k$, let

$$c_\lambda^{(k)} = \sum_{v\in\mathfrak{S}_\lambda} q^{\ell(v)} T_v.$$

Given a path $\mathfrak{t} \in \hat{\mathcal{H}}_k^\lambda$, write

$$d_{\mathfrak{t}}^{(k)} = d_{\lambda^{(k-1)}\to\lambda^{(k)}}^{(k)} d_{\lambda^{(k-2)}\to\lambda^{(k-1)}}^{(k-1)} \cdots d_{\lambda^{(0)}\to\lambda^{(1)}}^{(1)} \qquad \text{for} \qquad \mathfrak{t} = \big(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)}\big).$$

We will usually write $d_{\mathfrak{t}} = d_{\mathfrak{t}}^{(k)}$ for $\lambda \in \hat{\mathcal{H}}_k$ and $\mathfrak{t} \in \hat{\mathcal{H}}_k^\lambda$.

Theorem 5.1 (Murphy [Mu])**.** *If $i = 1, 2, \dots,$ the set*

$$\mathscr{H}_i = \Big\{ d_{\mathfrak{s}}^* c_\lambda^{(i)} d_{\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \hat{H}_i^\lambda,\ (\lambda, \ell) \in \hat{H}_i \Big\}$$

is an R–basis for $\mathcal{H}_i$. Moreover, the following statements hold:

(1) Given $\lambda \in \hat{H}_i$, $\mathfrak{t} \in \hat{H}_i^\lambda$, and $h \in \mathcal{H}_i$, there exist coefficients $r_{\mathfrak{u}} \in R$, for $\mathfrak{u} \in \hat{H}_i^\lambda$, such that for all $\mathfrak{s} \in \hat{H}_i^\lambda$,

$$d_{\mathfrak{s}}^* c_\lambda^{(i)} d_{\mathfrak{t}} h \equiv \sum_{\mathfrak{u}\in\hat{H}_i^\lambda} r_{\mathfrak{u}} d_{\mathfrak{s}}^* c_\lambda^{(i)} d_{\mathfrak{u}} \mod \mathcal{H}_i^{\rhd\lambda},$$

where $H_i^{\rhd\lambda}$, for $\lambda \in \hat{H}_i$, is the R–module freely generated by

$$\Big\{ d_{\mathfrak{s}}^* c_\mu^{(i)} d_{\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \hat{H}_i^\mu \text{ where } (mu \in \hat{H}_i \text{ and } \mu \rhd \lambda \Big\}.$$

(2) If $\lambda \in \hat{H}_i$, and $\mathfrak{s}, \mathfrak{t} \in \hat{H}_i^\lambda$, then $* : d_{\mathfrak{s}}^* c_\lambda^{(i)} d_{\mathfrak{t}} \mapsto d_{\mathfrak{t}}^* c_\lambda^{(i)} d_{\mathfrak{s}}$.

If $\lambda \in \hat{\mathcal{H}}_{i-1}$, $\mu \in \hat{\mathcal{H}}_i$ and $\mu = \lambda \cup \{(j, \mu_j)\}$, let $a_j = \sum_{r=1}^{j} \mu_r$, and define

$$u_{\lambda\to\mu}^{(i)} = T_{i,a_j} \sum_{r=0}^{\lambda_j} q^r T_{a_j, a_j - r}.$$

The next two statements respectively follow from Theorem 4.10 and Proposition 6.1 of [Mat].

Proposition 5.2. *Let $\lambda \in \hat{\mathcal{H}}_i$ and $\mu^{(1)}, \dots, \mu^{(p)}$ be an ordering of*

$$\{\mu \mid \mu \in \hat{\mathcal{H}}_{i+1} \text{ and } \lambda \to \mu\}$$

such that $\mu^{(s)} \rhd \mu^{(t)}$ whenever $t > s$. If

$$N_j = \sum_{s\leqslant j} \big(c_\lambda^{(i)} + \mathcal{H}_i^{\rhd\lambda}\big) \otimes u_{\lambda\to\mu}^{(i+1)} \mathcal{H}_{i+1} \qquad \text{for } j = 1, \dots, p,$$

then

$$\{0\} = N_0 \overset{\mu^{(1)}}{\subseteq} N_1 \overset{\mu^{(2)}}{\subseteq} \cdots \overset{\mu^{(p)}}{\subseteq} N_p = H_i^\lambda \otimes_{\mathcal{H}_i} \mathcal{H}_{i+1}$$

is a filtration by $\mathcal{H}_{i+1}$–modules where, for $j = 1, \dots, p$,

$$\big(c_{\mu^{(j)}}^{(i+1)} + H_{i+1}^{\rhd\mu^{(j)}}\big) \mapsto \big(c_\lambda^{(i)} + H_i^{\rhd\lambda}\big) \otimes u_{\lambda\to\mu^{(j)}}^{(i+1)} + N_{j-1}$$

under the isomorphism $H_{i+1}^{\mu^{(j)}} \cong N_j / N_{j-1}$.

Proposition 5.3. *Let $\mu \in \hat{\mathcal{H}}_{i+1}$ and $\lambda^{(1)}, \ldots, \lambda^{(r)}$ be an ordering of*

$$\{\lambda \mid \lambda \in \hat{\mathcal{H}}_i \text{ and } \lambda \to \mu\}$$

such that $\lambda^{(s)} \rhd \lambda^{(t)}$ whenever $t > s$. If

$$M_j = \sum_{s \leqslant j} (c_\mu^{(i+1)} + \mathcal{H}_{i+1}^{\rhd \mu}) d_{\lambda \to \mu}^{(i+1)} \mathcal{H}_i \qquad \text{for } j = 1, \ldots, r,$$

then

$$\{0\} = M_0 \overset{\lambda^{(1)}}{\subseteq} M_1 \overset{\lambda^{(2)}}{\subseteq} \cdots \overset{\lambda^{(r)}}{\subseteq} M_r = \mathrm{Res}_{\mathcal{H}_i}^{\mathcal{H}_{i+1}} (H_{i+1}^{\mu})$$

is a filtration by $\mathcal{H}_i$–modules where, for $j = 1, \ldots, r$,

$$(c_{\lambda^{(j)}}^{(i)} + H_i^{\rhd \lambda^{(j)}}) \mapsto (c_\lambda^{(i+1)} + H_{i+1}^{\rhd \mu}) d_{\lambda \to \mu^{(j)}}^{(i+1)} + M_{j-1}$$

under the isomorphism $H_i^{\lambda^{(j)}} \cong M_j / M_{j-1}$.

5.3. **Brauer algebras.** The Brauer algebras $B_k(z)$ were defined by Brauer [Br]. Wenzl [We] showed that the Brauer algebras are obtained from the group algebra of the symmetric group by the Jones basic construction, and that the Brauer algebras over a field of characteristic zero are generically semisimple. Cellularity of the Brauer algebras was established by Graham and Lehrer [GL]. A Murphy type cellular basis for the Brauer algebras has been given in [En1].

Let z be an indeterminant over $\mathbb{Z}$, and $R = \mathbb{Z}[z]$. Following §2.2 of [DRV], the Brauer algebra $B_k = B_k(z)$ is the unital R–algebra presented by the generators

$$t_u \quad (u \in \mathfrak{S}_k), \qquad \text{and} \qquad e_1, \ldots, e_{k-1},$$

and the relations

$$\begin{aligned}
&t_u t_v = t_{uv}, && t_{s_i} e_i = e_i t_{s_i} = e_i, && e_i t_{s_{i-1}} e_i = e_i, && e_i t_{s_{i+1}} e_i = e_i, \\
&e_i e_{i+1} e_i = e_i && e_i e_{i-1} e_i = e_i, && e_{i+1} e_i = e_{i+1} t_{s_i} t_{s_{i+1}}, && e_i e_{i+1} = t_{s_{i+1}} t_{s_i} e_{i+1}.
\end{aligned}$$

For $i, j, = 1, \ldots, k$, let

$$t_{i,j} = \begin{cases} t_{s_i} t_{s_{i+1}} \cdots t_{s_{j-1}}, & \text{if } i \leqslant j, \\ t_{s_{i-1}} t_{s_{i-2}} \cdots t_{s_j}, & \text{if } i > j. \end{cases}$$

The involution $* : B_k(z) \to B_k(z)$ given by

$$t_v \mapsto t_{v^{-1}} \qquad \text{and} \qquad e_i \mapsto e_i \qquad (1 \leqslant i < k)$$

is an algebra anti–automorphism of $B_k(z)$. The map

$$\begin{aligned}
B_k/(B_k e_{k-1} B_k) &\to R\mathfrak{S}_k, \\
t_v + (e_k) &\mapsto v, \qquad \text{for } v \in \mathfrak{S}_k,
\end{aligned}$$

is an algebra isomorphism. For $i = 0, 1, \ldots,$ let $H_i = R\mathfrak{S}_i$. Together with Theorem 5.1, the defining relations show that

$$R = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \qquad \text{and} \qquad R = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$$

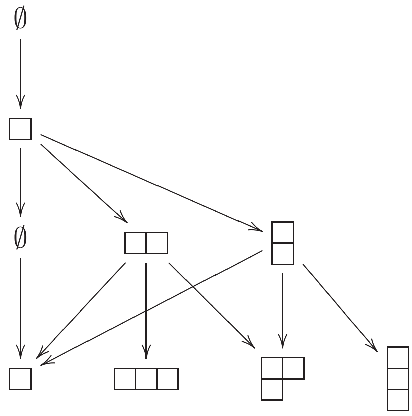
satisfy the axioms (A)–(H) above (*cf.* §5.2 of [GG] and Remark 2.4 of [DRV]). For $i = 0, 1, \ldots,$ let $\hat{H}_i = \{\lambda \mid \lambda \vdash i\}$. With the specialisation $q = 1$, the the Murphy basis given in §5.2 yields the simisimple branching diagram $\hat{H}$ for the group algebra of the symmetric group (5.7). For $i = 0, 1, \ldots,$ let

$$\hat{B}_i = \big\{(\lambda, \ell) \,\big|\, \lambda \in H_{i-2\ell}, \text{ for } \ell = 0, 1, \ldots, \lfloor i/2 \rfloor \big\}$$

and order $\hat{B}_i$ by writing $(\lambda, \ell) \trianglerighteq (\mu, m)$, for $(\lambda, \ell), (\mu, m) \in \hat{B}_i$, if either:

(1) $\ell > m$, or
(2) $\ell = m$ and $\lambda \trianglerighteq \mu$ in the dominance order on elements of $\hat{H}_{i-2\ell}$.

The first three levels of $\hat{B}$ are given in (5.8).



(5.8)

For $i = 2, 3, \ldots$, let

$$e_{i-1}^{(\ell)} = \underbrace{e_{i-2\ell+1}e_{i-2\ell+3}\cdots e_{i-1}}_{\ell \text{ factors}} \qquad \text{if } \ell = 0, 1, \ldots, \lfloor i/2 \rfloor,$$

and write

$$e_{i-1}^{(\ell)} = 0 \qquad \text{if } \ell > \lfloor i/2 \rfloor.$$

For $i = 0, 1, \ldots$, and $(\mu, m) \in \hat{A}_i$, let

$$x_{(\mu,m)}^{(i)} = c_\mu^{(i-2m)} e_{i-1}^{(m)} \qquad \text{where} \qquad c_\mu^{(i-2m)} = \sum_{v \in \mathfrak{S}_\mu} t_v.$$

If $\lambda \in \hat{H}_{i-2m-1}$ and $\mu \in \hat{H}_{i-2m}$, such that $\mu = \lambda \cup \{(j, \mu_j)\}$, let $a_j = \sum_{r=1}^{j} \mu_j$ and define

$$\bar{u}_{\lambda\to\mu}^{(i-2m)} = t_{i-2m,a_j} \sum_{r=0}^{\lambda_j} t_{a_j,a_j-r} \qquad \text{and} \qquad \bar{d}_{\lambda\to\mu}^{(i-2m)} = t_{a_j,i-2m}.$$

If $(\lambda, \ell) \in \hat{B}_i$, $(\mu, m) \in \hat{B}_{i+1}$, and $(\lambda, \ell) \to (\mu, m)$ in $\hat{B}$, define

$$b_{(\lambda,\ell)\to(\mu,m)}^{(i+1)} = \begin{cases} \bar{d}_{\lambda\to\mu}^{(k-2m+1)} e_{k-1}^{(m)}, & \text{if } \ell = m, \\ e_{k-1}^{(m-1)} \bar{u}_{\mu\to\lambda}^{(k-2m+2)}, & \text{if } \ell = m-1. \end{cases}$$

For $\mathfrak{t} = ((\lambda^{(0)}, \ell_0), (\lambda^{(1)}, \ell_1), \ldots, (\lambda^{(i)}, \ell_i)) \in \hat{B}_i^{(\lambda,\ell)}$, let

$$b_\mathfrak{t} = b_{(\lambda^{(i-1)},\ell_{i-1})\to(\lambda^{(i)},\ell_i)}^{(i)} b_{(\lambda^{(i-2)},\ell_{i-2})\to(\lambda^{(i-1)},\ell_{i-1})}^{(i-1)} \cdots b_{(\lambda^{(0)},\ell_0)\to(\lambda^{(1)},\ell_1)}^{(1)}.$$

From Theorem 4.1, we obtain

Theorem 5.4. *If $i = 1, 2, \ldots$, the set*

$$\mathscr{B}_i = \left\{ b_\mathfrak{s}^* x_{(\lambda,\ell)}^{(i)} b_\mathfrak{t} \,\middle|\, \mathfrak{s}, \mathfrak{t} \in \hat{B}_i^{(\lambda,\ell)},\ (\lambda,\ell) \in \hat{B}_i, \text{ and } \ell = 0, 1, \ldots \lfloor i/2 \rfloor \right\}, \tag{5.9}$$

is an R–basis for B_i, and $(B_i, *, \hat{B}_i, \trianglerighteq, \mathscr{B}_i)$ is a cell datum for B_i.

Remark 5.5. The basis (5.9) coincides with the Murphy–type basis for $B_i(z)$ given in [En1].

5.4. **Birman–Murakami–Wenzl algebras.** The BMW algebras $B_k(q,z)$ were defined by Birman and Wenzl [BW] and Murakami [Mur] to give an algebraic realisation of the Kauffman link invariant [Ka]. Wenzl [We] showed that the BMW algebras are obtained from the Iwahori–Hecke algebras of the symmetric group by a Jones basic construction, and that the BMW algebras over a field of characteristic zero are generically semisimple. Cellularity of the BMW algebras was proved by Xi [Xi1] and a Murphy–type cellular basis for the BMW algebras was given in [En1].

Let $R = \mathbb{Z}[q^{\pm 1}, z^{\pm 1}, (q-q^{-1})^{-1}]$, where q, z are indeterminants over $\mathbb{Z}$. Following §3 of [We] or §2.3 of [DRV], the BMW algebra $W_k = W_k(q,z)$ is the unital R–algebra presented by the generators $g_1, \dots, g_{k-1}$, which are assumed to be invertible, and relations

$$\begin{aligned}
&g_i g_j = g_j g_i, && i \neq j, j+1,\\
&g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, && i = 1, \dots, k-2,\\
&e_i g_{i-1}^{\pm 1} e_i = z^{\pm 1} e_i, && i = 2, \dots, k-1,\\
&g_i e_i = e_i g_i = z^{-1} g_i, && i = 1, \dots, k-1,
\end{aligned}$$

where e_i is defined by

$$g_i - g_i^{-1} = (q-q^{-1})(1-e_i).$$

The above relations also imply that

$$\begin{aligned}
&(g_i - q)(g_i + q^{-1})(g_i - r^{-1}) = 0,\\
&e_i^2 = \big(1 + \tfrac{r-r^{-1}}{q-q^{-1}}\big) e_i,\\
&e_{i+1} e_i = e_{i+1} g_i g_{i+1},\\
&e_i e_{i+1} = g_{i+1} g_i e_{i+1},\\
&e_i g_{i+1}^{\pm 1} e_i = z^{\pm 1} e_i,\\
&e_i e_j = e_j = e_j e_i, \quad \text{if } i \neq j+1.
\end{aligned}$$

If $v \in \mathfrak{S}_k$ and $v = s_{i_1} s_{i_2} \cdots s_{i_j}$ is a reduced expression then the element

$$g_v = g_{i_1} g_{i_2} \cdots g_{i_j}$$

is well defined. For $i, j = 1, 2, \dots,$ let

$$g_{i,j} = \begin{cases} g_i g_{i+1} \cdots g_{j-1}, & \text{if } j \geqslant i,\\ g_{i-1} g_{i-2} \cdots g_j, & \text{if } i > j. \end{cases}$$

The map $* : W_k(q,z) \to W_k(q,z)$ given by

$$g_v \mapsto g_{v^{-1}}, \qquad (v \in \mathfrak{S}_k) \qquad \text{and} \qquad e_i \mapsto e_i \qquad (1 \leqslant i < k),$$

is an algebra anti–automorphism of $W_k(q,z)$. The map

$$\begin{aligned}
W_k/(W_k e_{k-1} W_k) &\to \mathcal{H}_k = \mathcal{H}_k(q^2),\\
g_v + (e_{k-1}) &\mapsto T_v, && \text{for } v \in \mathfrak{S}_k,
\end{aligned}$$

is an algebra isomorphism. Together with Theorem 5.1, the defining relations relations and Remark 2.9 of [DRV] show that

$$R = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \qquad \text{and} \qquad R = \mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \cdots$$

satisfy the axioms (A)–(H) above (*cf.* §5.8 of [GG]). For $i = 0, 1, \dots,$ let

$$\widehat{W}_i = \Big\{ (\lambda, \ell) \,\Big|\, \lambda \in \hat{\mathcal{H}}_{i-2\ell}, \text{ for } \ell = 0, 1, \dots, \lfloor i/2 \rfloor \Big\}$$

and order $\widehat{W}_i$ by writing $(\lambda, \ell) \trianglerighteq (\mu, m)$, for $(\lambda, \ell), (\mu, m) \in \widehat{W}_i$, if either:

(1) $\ell > m$, or
(2) $\ell = m$ and $\lambda \trianglerighteq \mu$ as elements of $\lambda \in \hat{\mathcal{H}}_{i-2\ell}$.

The first three levels of $\widehat{W}$ are given in (5.8). For $i = 2, 3, \dots$, let

$$e_{i-1}^{(\ell)} = \underbrace{e_{i-2\ell+1}e_{i-2\ell+3}\cdots e_{i-1}}_{\ell \text{ factors}} \qquad \text{if } \ell = 0, 1, \dots, \lfloor i/2 \rfloor,$$

and write

$$e_{i-1}^{(\ell)} = 0 \qquad \text{if } \ell > \lfloor i/2 \rfloor.$$

For $i = 0, 1, \dots$, and $(\mu, m) \in \hat{A}_i$, let

$$x_{(\mu,m)}^{(i)} = \bar{c}_{\mu}^{(i-2m)} e_{i-1}^{(m)} \qquad \text{where} \qquad \bar{c}_{\mu}^{(i-2m)} = \sum_{v \in \mathfrak{S}_\mu} q^{\ell(v)} g_v.$$

Let $i = 1, 2, \dots$, and $\lambda \in \hat{\mathcal{H}}_{i-1}$ and $\mu \in \hat{\mathcal{H}}_i$. If $\mu = \lambda \cup \{(j, \mu_j)\}$, let $a_j = \sum_{r=1}^{j} \mu_j$, and define

$$\bar{u}_{\lambda\to\mu}^{(i)} = g_{i,a_j} \sum_{r=0}^{\lambda_j} q^r g_{a_j, a_j - r} \qquad \text{and} \qquad \bar{d}_{\lambda\to\mu}^{(i)} = g_{a_j,i}.$$

If $(\lambda, \ell) \in \widehat{W}_i$, $(\mu, m) \in \widehat{W}_{i+1}$, and $(\lambda, \ell) \to (\mu, m)$ in $\widehat{W}$, define

$$g_{(\lambda,\ell)\to(\mu,m)}^{(i+1)} = \begin{cases} \bar{d}_{\lambda\to\mu}^{(k-2m+1)} e_{k-1}^{(m)}, & \text{if } \ell = m, \\ e_{k-1}^{(m-1)} \bar{u}_{\mu\to\lambda}^{(k-2m+2)}, & \text{if } \ell = m-1. \end{cases}$$

For $\mathfrak{t} = ((\lambda^{(0)}, \ell_0), (\lambda^{(1)}, \ell_1), \dots, (\lambda^{(i)}, \ell_i)) \in \widehat{W}_i^{(\lambda,\ell)}$, let

$$g_{\mathfrak{t}} = g_{(\lambda^{(i-1)},\ell_{i-1})\to(\lambda^{(i)},\ell_i)}^{(i)} g_{(\lambda^{(i-2)},\ell_{i-2})\to(\lambda^{(i-1)},\ell_{i-1})}^{(i-1)} \cdots g_{(\lambda^{(0)},\ell_0)\to(\lambda^{(1)},\ell_1)}^{(1)}.$$

From Theorem 4.1 we obtain:

Theorem 5.6. *If $i = 1, 2, \dots$, the set*

$$\mathscr{W}_i = \left\{ g_{\mathfrak{s}}^* x_{(\lambda,\ell)}^{(i)} g_{\mathfrak{t}} \,\middle|\, \mathfrak{s}, \mathfrak{t} \in \widehat{W}_i^{(\lambda,\ell)},\ (\lambda,\ell) \in \widehat{W}_i, \text{ and } \ell = 0, 1, \dots \lfloor i/2 \rfloor \right\}, \tag{5.10}$$

is an R–basis for W_i, and $(W_i, *, \widehat{W}_i, \trianglerighteq, \mathscr{W}_i)$ is a cell datum for W_i.

Remark 5.7. The basis (5.10) differs from the Murphy–type basis for $W_i(q,z)$ given in [En1] by a unitriangular transformation.

5.5. **Temperley–Lieb algebras.** The Temperley–Lieb algebras were defined by Jones [Jo], who used them to define link invariants in [Jo1]. The cellularity of Temperley–Lieb algebras was established by Graham and Lehrer [GL]. Härterich [Hä] has given Murphy bases for generalised Temperley–Lieb algebras, and Goodman and Graber [GG1] have shown that the Temperley–Lieb algebras form a strongly coherent tower of cellular algebras.

Let z be an indeterminant and $R = \mathbb{Z}[z]$. The Temperley–Lieb algebra $A_k = A_k(z)$ is the unital R–algebra presented by the generators $e_1, \dots, e_{k-1}$ and the relations

$$\begin{aligned} e_i e_{i-1} e_i &= e_i, && i = 2, \dots, k-1, \\ e_{i-1} e_i e_{i-1} &= e_{i-1}, && i = 2, \dots, k-1, \\ e_i^2 &= z e_i, && i = 1, \dots, k-1, \\ e_i e_j &= e_j e_i, && i \neq j+1. \end{aligned}$$

The involution $A_k \to A_k$ given by

$$e_i \mapsto e_i \qquad (\text{for } 1 \leqslant i < k)$$

is an algebra anti–automorphism and the map

$$\begin{aligned} A_k/(A_k e_{k-1} A_k) &\to R \\ 1_{A_k} + (e_{k-1}) &\mapsto 1_R, \end{aligned}$$

is an algebra isomorphism. The defining relations show that

$$R = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \qquad \text{and} \qquad R = H_0 = H_1 = H_2 = \cdots$$

satisfy the axioms (A)–(H) above (*cf.* §5.3 of [GG]). For $i = 0, 1, \ldots,$ let

$$\hat{A}_i = \{i - 2\ell \mid \ell = 0, 1, \ldots, \lfloor i/2 \rfloor\}$$

and order $\hat{A}_i$ by writing $(i - 2\ell) \trianglerighteq (i - 2m)$ if $\ell \geqslant m$ as integers. The branching diagram $\hat{A}$ has

(1) vertices on level i: $\{i - 2\ell \mid \ell = 0, 1, \ldots, \lfloor i/2 \rfloor\}$ and

(2) an edge $i - 2\ell \to i + 1 - 2m$ if either $m = \ell$ or $m = \ell + 1$.

The first few levels of $\hat{A}$ are given in (5.11).

$$\begin{tikzcd}
0 & \emptyset \arrow[d] & & \\
 & 1 \arrow[d] \arrow[dr] & & \\
2 & 0 \arrow[d] & 2 \arrow[dl] \arrow[d] & \\
 & 1 \arrow[d] \arrow[dr] & 3 \arrow[d] \arrow[dr] & \\
4 & 0 \arrow[d] & 2 \arrow[dl] \arrow[d] & 4 \arrow[dl] \arrow[d] \\
 & 1 & 3 & 5
\end{tikzcd} \tag{5.11}$$

For $i, \ell = 2, 3, \ldots,$ let

$$e_{i-1}^{(\ell)} = \underbrace{e_{i-2\ell+1} e_{i-2\ell+3} \cdots e_{i-1}}_{\ell \text{ factors}} \qquad \text{if } \ell = 0, 1, \ldots, \lfloor i/2 \rfloor,$$

and write

$$e_{i-1}^{(\ell)} = 0 \qquad \text{if } \ell > \lfloor i/2 \rfloor.$$

For $i = 0, 1, \ldots,$ and $(i - 2\ell) \in \hat{A}_i$, let

$$x_{i-2\ell}^{(i)} = e_{i-1}^{(\ell)},$$

and, if $(i - 2\ell) \to (i + 1 - 2m)$ in the diagram $\hat{A}$, let

$$a_{(i-2\ell)\to(i+1-2m)}^{(i+1)} = \begin{cases} e_{i-1}^{(m)}, & \text{if } \ell = m, \\ e_{i-1}^{(m-1)}, & \text{if } \ell = m - 1. \end{cases}$$

For $\mathfrak{t} = (0, 1, 2 - 2\ell_2, \ldots, i - 2\ell_i) \in \hat{A}_i^{(i-2\ell_i)}$, let

$$a_{\mathfrak{t}} = a_{(i-1-2\ell_{i-1})\to(i-2\ell_i)}^{(i)} a_{(i-2-2\ell_{i-2})\to(i-1-2\ell_{i-1})}^{(i-1)} \cdots a_{0\to 1}^{(1)}.$$

From Theorem 4.1, we obtain:

Theorem 5.8. *If $i = 1, 2, \dots,$ the set*

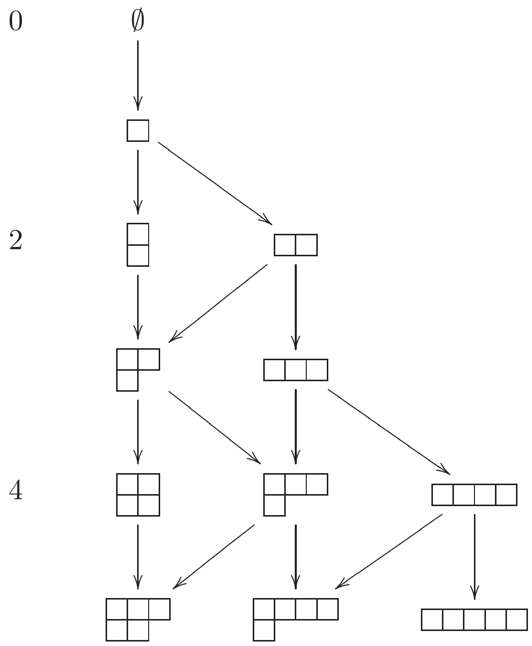
$$\mathscr{A}_i = \left\{ a_{\mathfrak{s}}^* x_{i-2\ell}^{(i)} a_{\mathfrak{t}} \,\middle|\, \mathfrak{s}, \mathfrak{t} \in \hat{A}_i^{(i-2\ell)},\ (i-2\ell) \in \hat{A}_i, \text{ and } \ell = 0, 1, \dots \lfloor i/2 \rfloor \right\},$$

is an R–basis for A_i, and $(A_i, *, \hat{A}_i, \trianglerighteq, \mathscr{A}_i)$ is a cell datum for A_i.

Remark 5.9. In the literature, it is usual to write

$$\hat{A}_i = \{\lambda \mid \lambda \text{ is a partition of } i \text{ with at most two non–zero parts}\}$$

so that the branching diagram (5.11) is represented as



5.6. **Partition algebras.** The partition algebras $\mathcal{A}_k(n)$, for $k, n \in \mathbb{Z}_{\geqslant 0}$, are a family of algebras defined in the work of Martin and Jones in [Mar], [Mar2], [Jo2] in connection with the Potts model and higher dimensional statistical mechanics. By [Jo2], the partition algebra $\mathcal{A}_k(n)$ is in Schur–Weyl duality with the symmetric group $\mathfrak{S}_n$ acting diagonally on the k–fold tensor product $V^{\otimes k}$ of its n–dimensional permutation representation V. In [Mar3], Martin defined the partition algebras $\mathcal{A}_{k+\frac{1}{2}}(n)$ as the centralisers of the subgroup $\mathfrak{S}_{n-1} \subseteq \mathfrak{S}_n$ acting on $V^{\otimes k}$. Including the algebras $\mathcal{A}_{k+\frac{1}{2}}(n)$ in the tower

$$\mathcal{A}_0(n) \subseteq \mathcal{A}_{\frac{1}{2}}(n) \subseteq \mathcal{A}_1(n) \subseteq \mathcal{A}_{1+\frac{1}{2}}(n) \subseteq \cdots \tag{5.12}$$

allowed for the simultaneous analysis of the whole tower of algebras (5.12) using the Jones Basic construction by Martin [Mar3] and Halverson and Ram [HR].

A presentation for the partition algebras has been given by Halverson and Ram [HR] and East [Ea] as follows.

Let z be an indeterminant over $\mathbb{Z}$ and $R = \mathbb{Z}[z]$. If $k \in \mathbb{Z}_{\geqslant 0}$, the partition algebra $\mathcal{A}_k(z)$ is the unital associative R–algebra presented by the generators

$$t_{s_1}, \dots, t_{s_{k-1}}, p_1, p_{1+\frac{1}{2}}, p_2, \dots, p_k,$$

and the relations

(1) (Coxeter relations)
  (i) $t_{s_i}^2 = 1$, for $i = 1, \dots, k-1$.
  (ii) $t_{s_i} t_{s_j} = t_{s_j} t_{s_i}$, if $j \neq i + 1$.
  (iii) $t_{s_i} t_{s_{i+1}} t_{s_i} = t_{s_{i+1}} t_{s_i} t_{s_{i+1}}$, for $i = 1, \dots, k-2$.

(2) (Idempotent relations)
  (i) $p_i^2 = zp_i$, for $i = 1, \ldots, k$.
  (ii) $p_{i+\frac{1}{2}}^2 = p_{i+\frac{1}{2}}$, for $i = 1, \ldots, k-1$.
  (iii) $t_{s_i} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} s_i = p_{i+\frac{1}{2}}$, for $i = 1, \ldots, k-1$.
  (iv) $t_{s_i} p_i p_{i+1} = p_i p_{i+1} t_{s_i} = p_i p_{i+1}$, for $i = 1, \ldots, k-1$.

(3) (Commutation relations)
  (i) $p_i p_j = p_j p_i$, for $i = 1, \ldots k$ and $j = 1, \ldots, k$.
  (ii) $p_{i+\frac{1}{2}} p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_{i+\frac{1}{2}}$, for $i = 1, \ldots, k-1$ and $j = 1, \ldots, k-1$.
  (iii) $p_i p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_i$, for $j \neq i, i+1$.
  (iv) $t_{s_i} p_j = p_j t_{s_i}$, for $j \neq i, i+1$.
  (v) $t_{s_i} p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} t_{s_i}$, for $j \neq i-1, i+1$.
  (vi) $t_{s_i} p_i t_{s_i} = p_{i+1}$, for $i = 1, \ldots, k-1$.
  (vii) $t_{s_i} p_{i-\frac{1}{2}} t_{s_i} = t_{s_{i-1}} p_{i+\frac{1}{2}} t_{s_{i-1}}$, for $i = 2, \ldots, k-1$.

(4) (Contraction relations)
  (i) $p_{i+\frac{1}{2}} p_j p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$, for $j = i, i+1$.
  (ii) $p_i p_{j-\frac{1}{2}} p_i = p_i$, for $j = i, i+1$.

The above relations also imply that:

$$\begin{aligned}
&p_{i+\frac{1}{2}} t_{s_{i\pm 1}} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_{i\pm\frac{1}{2}}, \\
&p_i t_{s_i} p_i = p_i p_{i+1} = p_{i+1} t_{s_i} p_{i+1}, \\
&p_i p_{i+\frac{1}{2}} p_{i+1} = p_i t_{s_i}, \\
&p_{i+1} p_{i+\frac{1}{2}} p_i = p_{i+1} t_{s_i}.
\end{aligned}$$

The partition algebra $\mathcal{A}_{k-\frac{1}{2}}(z)$ is defined to be the subalgebra of $\mathcal{A}_k(z)$ generated by

$$t_{s_1}, \ldots, t_{s_{k-2}}, p_1, p_{1+\frac{1}{2}}, p_2, \ldots, p_{k-\frac{1}{2}}.$$

If $v \in \mathfrak{S}_k$ and $v = s_{i_1} \cdots s_{i_j}$ is a reduced expression, then $t_v = t_{s_{i_1}} \cdots t_{s_{i_j}}$ is well defined. For $i, j, = 1, \ldots, k$, let

$$t_{i,j} = \begin{cases} t_{s_i} t_{s_{i+1}} \cdots t_{s_{j-1}}, & \text{if } i \leqslant j, \\ t_{s_{i-1}} t_{s_{i-2}} \cdots t_{s_j}, & \text{if } i > j. \end{cases}$$

Following §5.7 of [GG], define the tower of algebras

$$R = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \qquad \text{and} \qquad R = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$$

by

$$A_{2i} = \mathcal{A}_i(z) \qquad \text{and} \qquad H_{2i} = R\mathfrak{S}_i, \qquad \text{for } i = 0, 1, \ldots,$$

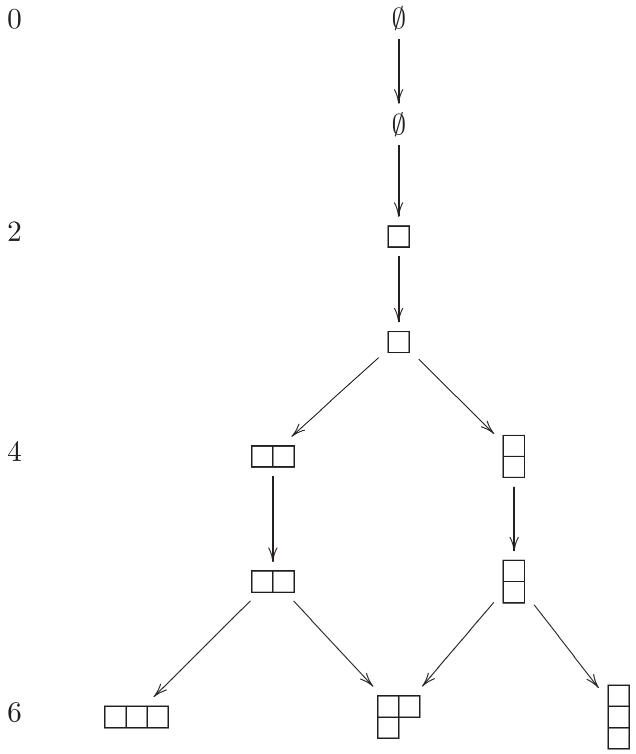
and

$$A_{2i+1} = \mathcal{A}_{i+\frac{1}{2}}(z) \qquad \text{and} \qquad H_{2i+1} = R\mathfrak{S}_i, \qquad \text{for } i = 0, 1, \ldots,$$

The branching diagram of the tower $R = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ is the graph $\hat{H}$ with

(1) vertices on level i: $\hat{H}_i$, and
(2) an edge $\lambda \to \mu$ in $\hat{H}$ if
 (a) $\lambda \in \hat{H}_{2i-1}$, $\mu \in \hat{H}_{2i}$ and $\lambda \subseteq \mu$, or
 (b) $\lambda \in \hat{H}_{2i}$, $\mu \in \hat{H}_{2i+1}$ and $\lambda = \mu$.

The first few levels of $\hat{H}$ are given in (5.13).



(5.13)

Let

$$p_i = e_{2i-1} \in A_{2i} \qquad \text{and} \qquad p_{i+\frac{1}{2}} = e_{2i} \in A_{2i+1} \qquad \text{for } i = 1, 2, \dots .$$

The map $* : A_{2i+1} \to A_{2i+1}$ given by

$$t_v \mapsto t_{v^{-1}} \quad (v \in \mathfrak{S}_i) \qquad \text{and} \qquad e_j \mapsto e_j \quad (1 \leqslant j \leqslant 2i)$$

is an algebra anti–automorphism of A_{2i+1}, and the restriction of $*$ to A_{2i} an algebra anti–automorphism of A_{2i}. The maps

$$\begin{aligned} A_{2i}/(A_{2i}e_{2i-1}A_{2i}) &\to H_i, \\ t_v + (e_{2i-1}) &\mapsto v, \qquad \text{for } v \in \mathfrak{S}_i, \end{aligned}$$

and

$$\begin{aligned} A_{2i+1}/(A_{2i+1}e_{2i}A_{2i+1}) &\to H_i, \\ t_v + (e_{2i}) &\mapsto v, \qquad \text{for } v \in \mathfrak{S}_i, \end{aligned}$$

are algebra isomorphisms.

Together with Theorem 5.1, the defining relations relations (2)(i)–(4)(ii) show that

$$R = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \qquad \text{and} \qquad R = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$$

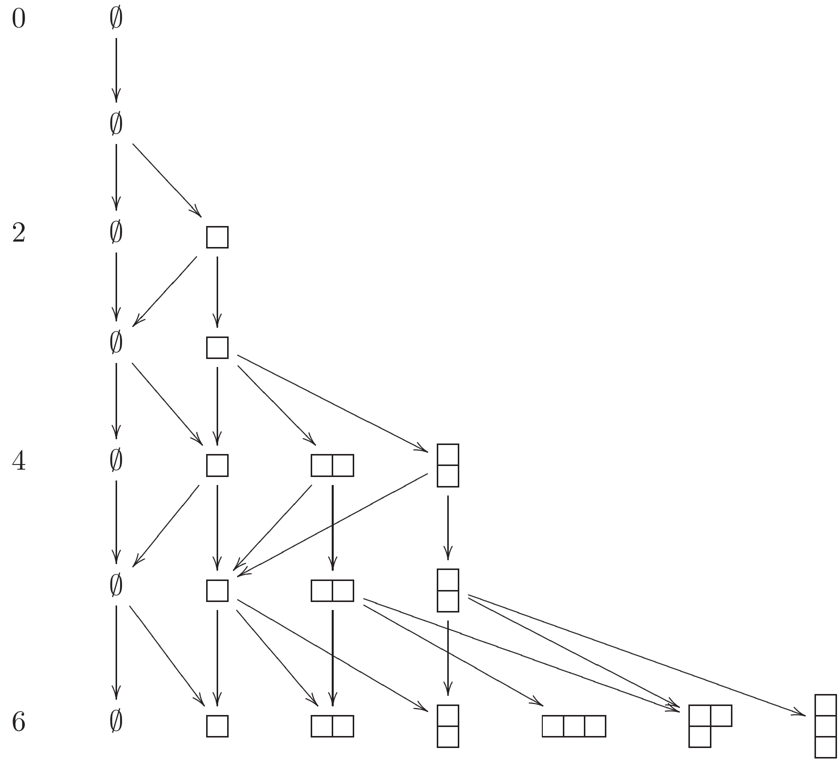
satisfy the axioms (A)–(H) above (*cf.* §5.7 of [GG]). For $i = 0, 1, \dots,$ let

$$\hat{A}_i = \Big\{ (\lambda, \ell) \,\Big|\, \lambda \in \hat{H}_{i-2\ell}, \text{ for } \ell = 0, 1, \dots, \lfloor i/2 \rfloor \Big\}$$

and order $\hat{A}_i$ by writing $(\lambda, \ell) \trianglerighteq (\mu, m)$, for $(\lambda, \ell), (\mu, m) \in \hat{A}_i$, if either:

(1) $\ell > m$, or
(2) $\ell = m$ and $\lambda \trianglerighteq \mu$ as elements of $H_{i-2\ell}$.

The first few levels of $\hat{A}$ are given in (5.14).



(5.14)

For $i = 2, 3, \ldots,$ let

$$e_{i-1}^{(\ell)} = \underbrace{e_{i-2\ell+1} e_{i-2\ell+3} \cdots e_{i-1}}_{\ell \text{ factors}} \qquad \text{if } \ell = 0, 1, \ldots, \lfloor i/2 \rfloor,$$

and write

$$e_{i-1}^{(\ell)} = 0 \qquad \text{if } \ell > \lfloor i/2 \rfloor.$$

For $i = 0, 1, \ldots,$ and $(\mu, m) \in \hat{A}_i$, let

$$x_{(\mu,m)}^{(i)} = \bar{c}_\mu^{(i-2m)} e_{i-1}^{(m)} \qquad \text{where} \qquad \bar{c}_\mu^{(i-2m)} = \sum_{v \in \mathfrak{S}_\mu} t_v.$$

If $\lambda \in \hat{H}_{2i-1}$ and $\mu \in \hat{H}_{2i}$, such that $\mu = \lambda \cup \{(j, \mu_j)\}$, let $a_j = \sum_{r=1}^{j} \mu_j$ and define

$$\bar{u}_{\lambda \to \mu}^{(2i)} = t_{i,a_j} \sum_{r=0}^{\lambda_j} t_{a_j, a_j - r} \qquad \text{and} \qquad \bar{d}_{\lambda \to \mu}^{(2i)} = t_{a_j, i}.$$

If $\mu \in \hat{H}_{2i}$ and $\nu \in \hat{H}_{2i+1}$ such that $\mu \to \nu$ in $\hat{H}$, define

$$\bar{u}_{\mu \to \nu}^{(2i+1)} = \bar{d}_{\mu \to \nu}^{(2i+1)} = 1.$$

If $(\lambda, \ell) \in \hat{A}_{2i-1}$ and $(\mu, m) \in \hat{A}_{2i}$ and $(\lambda, \ell) \to (\mu, m)$, then

$$a_{(\lambda,\ell) \to (\mu,m)}^{(2i)} = \begin{cases} \bar{d}_{\lambda \to \mu}^{(2i-2m)} e_{2i-2}^{(m)}, & \text{if } \ell = m, \\ e_{2i-2}^{(m-1)}, & \text{if } \ell = m - 1; \end{cases}$$

and similarly, if $(\mu, m) \in \hat{A}_{2i}$ and $(\nu, n) \in \hat{A}_{2i+1}$ and $(\mu, m) \to (\nu, n)$, then

$$a_{(\mu,m) \to (\nu,n)}^{(2i+1)} = \begin{cases} e_{2i-1}^{(n)}, & \text{if } m = n, \\ e_{2i-1}^{(n-1)} \bar{u}_{\mu \to \nu}^{(2i-2n+2)}, & \text{if } m = n - 1. \end{cases}$$

For $\mathfrak{t} = ((\lambda^{(0)}, \ell_0), (\lambda^{(1)}, \ell_1), \ldots, (\lambda^{(i)}, \ell_i)) \in \hat{A}_i^{(\lambda,\ell)}$, let

$$a_{\mathfrak{t}} = a^{(i)}_{(\lambda^{(i-1)},\ell_{i-1})\to(\lambda^{(i)},\ell_i)} a^{(i-1)}_{(\lambda^{(i-2)},\ell_{i-2})\to(\lambda^{(i-1)},\ell_{i-1})} \cdots a^{(1)}_{(\lambda^{(0)},\ell_0)\to(\lambda^{(1)},\ell_1)}.$$

From Theorem 4.1, we obtain the following new cellular bases for the partition algebras.

Theorem 5.10. *If $i = 1, 2, \ldots,$ the set*

$$\mathscr{A}_i = \left\{ a^*_{\mathfrak{s}} x^{(i)}_{(\lambda,\ell)} a_{\mathfrak{t}} \,\middle|\, \mathfrak{s}, \mathfrak{t} \in \hat{A}_i^{(\lambda,\ell)},\ (\lambda,\ell) \in \hat{A}_i, \text{ and } \ell = 0, 1, \ldots \lfloor i/2 \rfloor \right\},$$

is an R–basis for A_i, and $(A_i, *, \hat{A}_i, \trianglerighteq, \mathscr{A}_i)$ is a cell datum for A_i.

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